CONTROL SYSTEMS NOTES

AKSHANSH CHAUDHARY

Control Systems Notes, First Edition

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Presented by: Akshansh Chaudhary
Graduate of BITS Pilani, Dubai Campus
Batch of 2011

Course content by: Dr. R. Gomathi Bhavani
Then Faculty, BITS Pilani, Dubai Campus

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Control Concept

CONTROL SYSTEM

one of the classific^{ns}

Open Loop CS

Closed loop CS
(involves feedback)

eg. a driver driving a car

force → Accl^{n} pedal → Rate of fuel → engine → speed

Reference i/p or Cmd signal → terms to refer to i/p signal

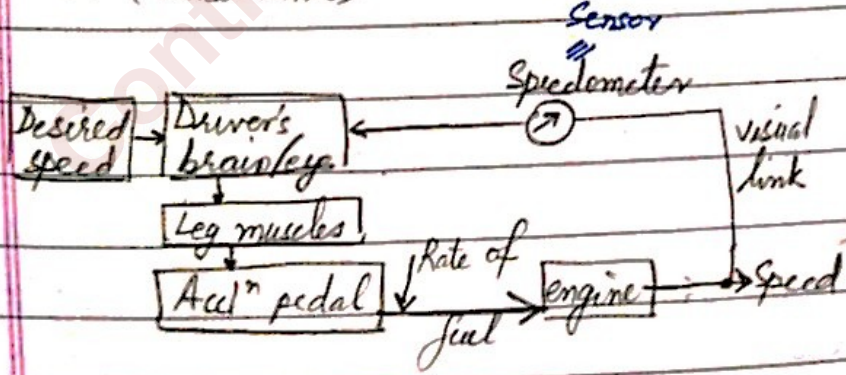
o/p variable or Controlled variable → terms to refer to o/p signal.

To make it a closed loop CS

eg

- If car needs to be driven at constt. speed, DESIRED speed is reference i/p going to brain → muscles → accl^{n} pedal
Also, the speed o/p var. is connected to brain (visual link)

When brain sees speedometer, it checks the difference from REFERENCE i/p to keep speed constt. This difference b/w reference i/p & feedback signal is called Error signal.

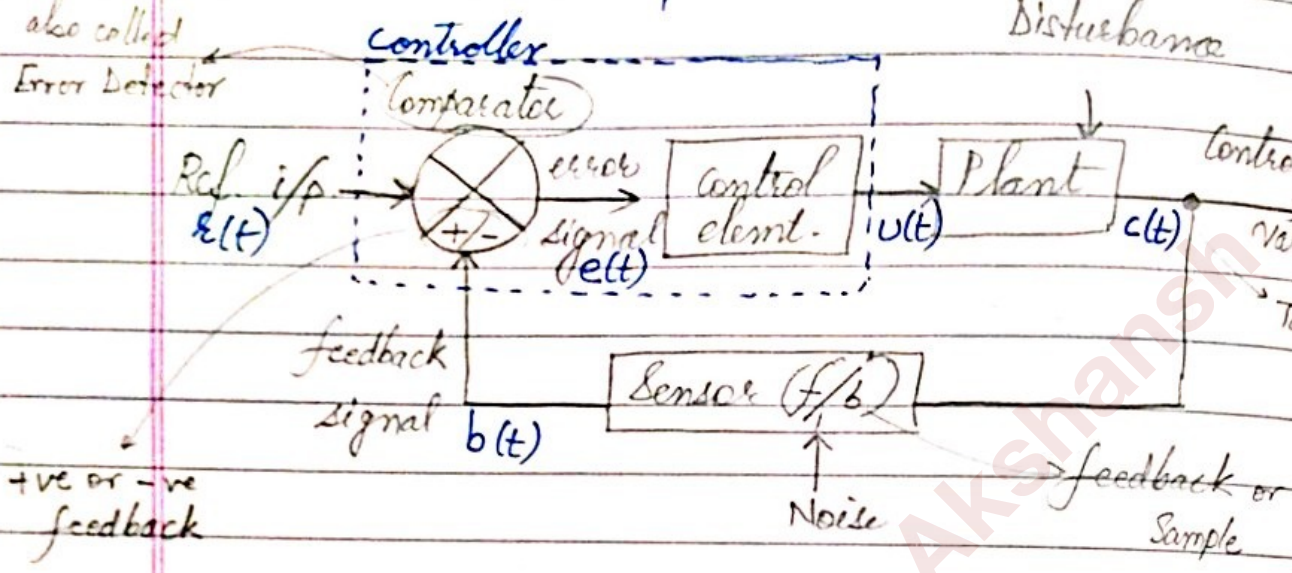


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* Disadvantage of fb control: Noise of sensor

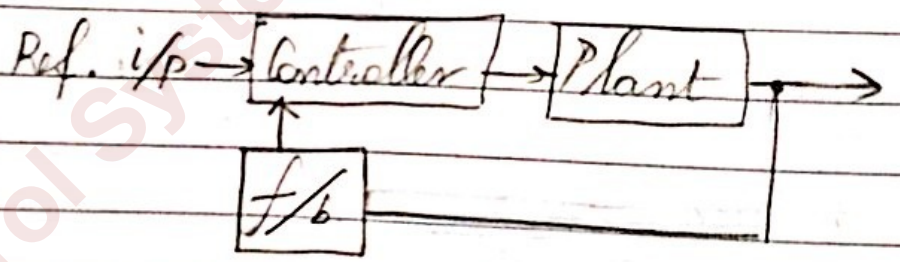
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• Generic CS closed loop

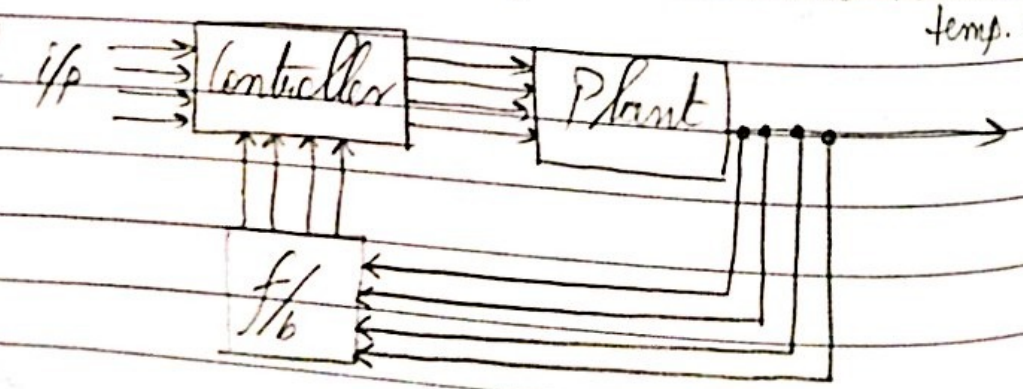


• "Time - "variant" : parameters depend on time mostly
• in previous ex., driver's brain/eye, accl" pedal, ... all are control elemts & the engine is plant

* SISO (Single i/p Single o/p) ~~loop~~ CS



* MIMO (Multiple i/p Multiple o/p) CS.
eg: In chemical plants, ∃ multiple variables like Pressure, flow, temp. etc



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KCL → nodal eqns
KVL → mesh eqns
Laws

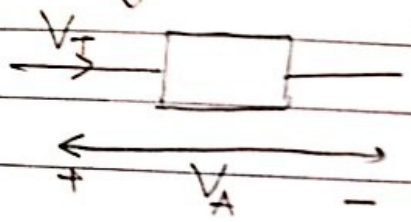
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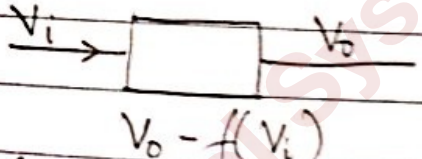
Chapter - 2
MATHEMATICAL MODELING OF PHYSICAL SYSTEM

Ideal elements

- Through variable (V_T) : 1 pt. of reference. → eg. Current.
- Across variable (V_A) : 2 pts. of reference → eg. Voltage : always measured across 2 terminals.

Building block





✓ We need
- Variables
- Apply laws using var } Mathematical model ✓
- Make eqns

mainly for linear time variant

- For SISO CS : Transfer f^n modelling
  MIMO CS : State space represent^n (or Vector Matrix)

* Mathematical modelling will invariably be of the form of differential equations for most physical sys. If the coefficients of DE are constts, then, model is said to be of Linear time invariant.

If the coefficients are f^ns of time, then model is Linear time varying model.

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Friction

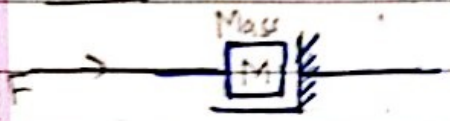
Viscous f | Coulombic f | Static f or Stiction

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✱ MECHANICAL Sys.

Translational

Mass

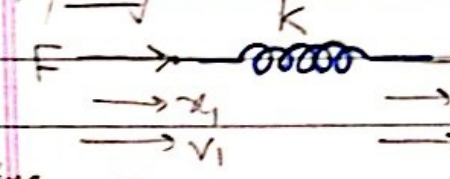


E storing element

$F = Ma$

$= M\frac{d^2x}{dt^2}$

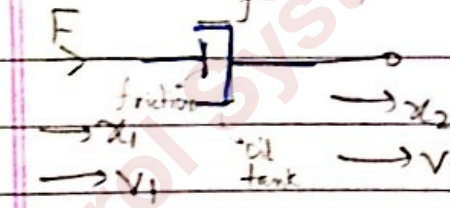
Spring



E storing element

$F = k(x_1 - x_2)$

Friction (or Damper) (B)



Viscous friction seen

E dissipating element

$F = f\frac{d}{dt}(x_1 - x_2)$

→ can also write (B)

Dashpot :- A device that introduces viscous f.

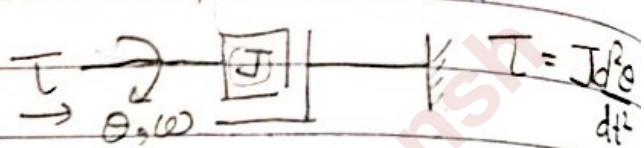
✓ K.E $= \frac{1}{2}MV^2$

✓ P.E $= \frac{1}{2}Kx^2$

✓ Power dissipⁿ :- fv^2

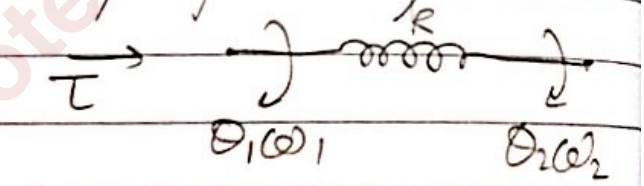
Rotnal Sys.

inertia



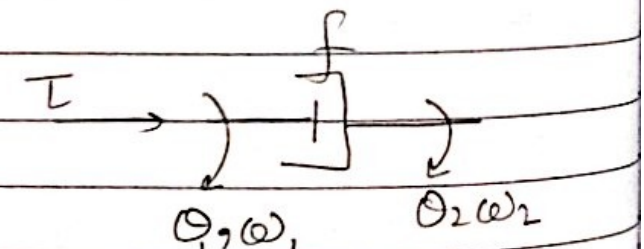
$\tau = J\frac{d^2\theta}{dt^2}$

Spring



$\tau = k(\theta_1 - \theta_2)$

Friction



$\tau = f\frac{d}{dt}(\theta_1 - \theta_2)$

KE $= \frac{1}{2}J\omega^2$

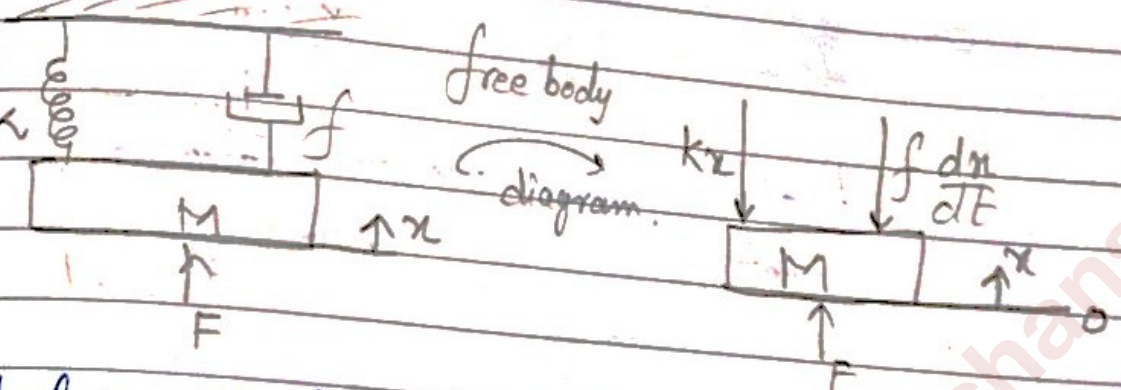
P.E $= \frac{1}{2}k\theta^2$

Power dissipⁿ :- $f\omega^2$

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✱ A mass spring - dashpot System (Translational)



Applying Newton's Law of motion, (force balance)

$$F - f\frac{dx}{dt} - kx = M\frac{d^2x}{dt^2} \quad (\underline{OR}) \quad F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx$$

→ linear
→ const. coeff.
→ 2nd order D.E

✱ Transfer functions (TF)

The TF of a linear time invariant sys is defined as the ratio of LT (Laplace Transform) of o/p variable to LT of i/p variable under the assumption that all initial conds are zero.

For a mass-spring dashpot sys.,

$$F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx \longrightarrow (1)$$

From (2) (PTO),

$$\mathcal{L}\left(M\frac{d^2x}{dt^2}\right) = M\left[\mathcal{L}\left(\frac{d^2x}{dt^2}\right)\right] = Ms^2 X(s)$$

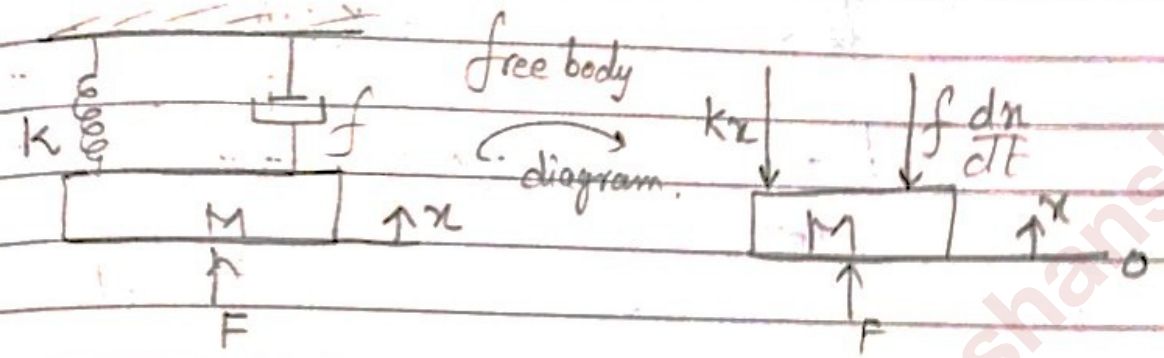
$(\because$ its 0 at $t=0)$

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D.E : Differential eq^n

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✱ A mass spring - dashpot System (Translational)



Applying Newton's Law of motion, (force balance)

$$F - f\frac{dx}{dt} - kx = M\frac{d^2x}{dt^2} \quad (\text{OR}) \quad \boxed{F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx}$$

→ linear
→ const. coeff.
→ 2nd order D.E

§ Transfer functions (TF)

The TF of a linear time invariant sys is defined as the ratio of LT (Laplace Transform^n) of o/p variable to LT of i/p variable under the assumption that all initial cond^ns are zero.

For a mass-spring dashpot sys.,

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From (2) (PTO),

$$\mathcal{L}\left(M\frac{d^2x}{dt^2}\right) = M\left[\mathcal{L}\left(\frac{d^2x}{dt^2}\right)\right]_{t=0} = Ms^2 X(s)$$

($\because$ its 0 at $t=0$)

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Note :- Laplace Transform^n rules :-

- $\mathcal{L}\left[\frac{dx}{dt}\right] = sX(s) - X(0)$

- $\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = s^2X(s) - sX(0) - X'(0)$ ⟶ (2)

So, By applying LT on (1) both sides, we get

$$F(s) = Ms^2X(s) + fsX(s) + kX(s)$$

$$T.F = G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + fs + k}$$

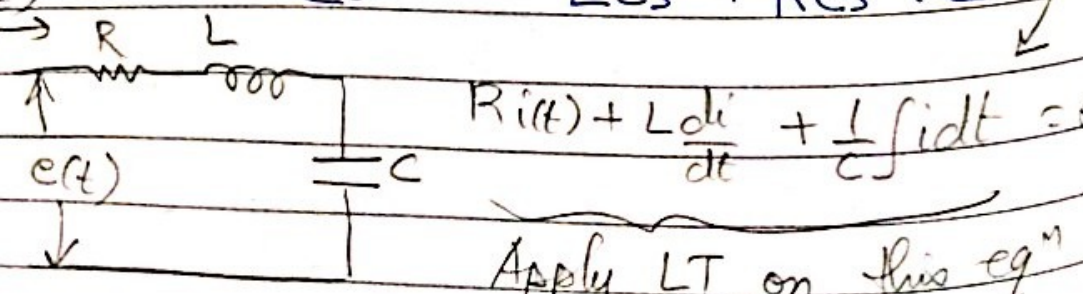
2nd order system.

An RLC series circuit has,

$$sLI(s) + RI(s) + \frac{1}{s}\frac{I_s}{C} = E(s)$$

capacitor voltage $\frac{1}{s}\frac{I(s)}{C} = E_o(s)$

$$T.F = \frac{E_o(s)}{E(s)} = \frac{1}{LCs^2 + RCs + 1}$$



$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e$$ (3)

Apply LT on this eq^n to get that eq^n.

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Small letter $x(t) \xrightarrow{LT} X(s)$ → capital letter

Doubts

- **TF**
  - ✓ relates i/p & o/p of linear time invariant sys.
  - ✓ doesn't give any info. about internal structure of sys.
  - ✓ with each block described as TF, convenient for block diagram representation.

We know

$$q = \int i\,dt \;;\; \frac{dq}{dt} = i \;;\; \frac{d^2q}{dt^2} = \frac{di}{dt}$$

↳ (4)

Using (4) in (3)

$$R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{1}{C}(q) = e \longrightarrow (5)$$

Comparing eqns (1) & (5) :-

also do for Rotational Mech. Sys.

Force-Voltage Analogy

Translational Mech. Sys. — RLC series ckt.

1)	Force (F)	is analogous to	Voltage (e)
2)	M (mass)		L (Inductance)
3)	f (friction)		R (Resistance)
4)	k (spring const.)		$\frac{1}{C}$ (reciprocal capacitance)
5)	x (displacement)		q (charge)
6)	$\frac{dx}{dt}$ (velocity)		$\frac{dq}{dt} = i$ (current)

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Misc:- $e = L\frac{di}{dt}$

Proof :- $\frac{1}{L}\int e\,dt = i_L$

Translational Mech. Sys. RLC parallel circuit.

$$M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx = F \quad \rightarrow (1)$$



By KCL on given circuit (A)

$$i = i_R + i_C + i_L$$

$$\frac{e}{R} \quad C\frac{de}{dt} \quad \frac{1}{L}\int e\,dt$$

$$C\frac{de}{dt} + \frac{1}{L}\int e\,dt + \frac{1}{R}e = i$$

Now, $e = \frac{d\phi}{dt}$, $\phi = \int e\,dt$

$$\frac{de}{dt} = \frac{d^2\phi}{dt^2}$$

$$\Rightarrow C\left(\frac{d^2\phi}{dt^2}\right) + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi = i \quad \rightarrow (2)$$

Comparing (1) & (2),

* Force - Current Analogy

Force	Current
F	i
M	C
K	1/L
f	1/R
x	ϕ
v	$d\phi/dt = e$

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Misc. Proof

$q = ce$ or $e = \frac{1}{c} q$ or $de = \frac{1}{c} dq$

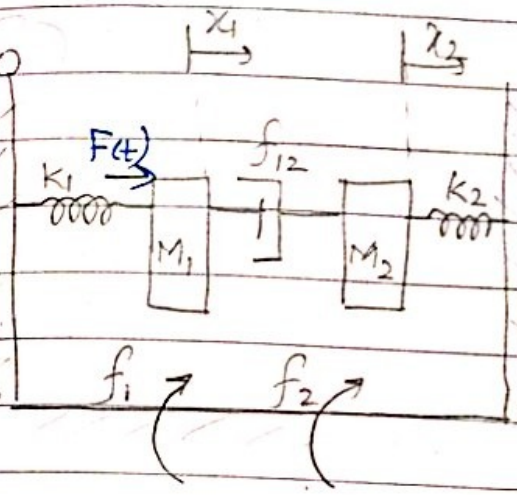
$\therefore \Rightarrow de = \frac{1}{c}(i\,dt) \Rightarrow \frac{de}{dt} = \frac{i_c}{c} \Rightarrow i_c = c\frac{de}{dt}$

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Q Obtain the transfer f^n for the mechanical sys. shown in the fig relating the o/p displacement x_2 with applied force $F(t)$.



Ans:- Find:- $\frac{x_2(s)}{F(s)}$

Method:- Analogous to Nodal method done before

Identifying nodes:-

No. of nodes = no. of displacements.

Generally, nodes are mass elements. In some cases, nodes may also be the interjunction (or intersection) b/w f & k or the free end where the forces apply.

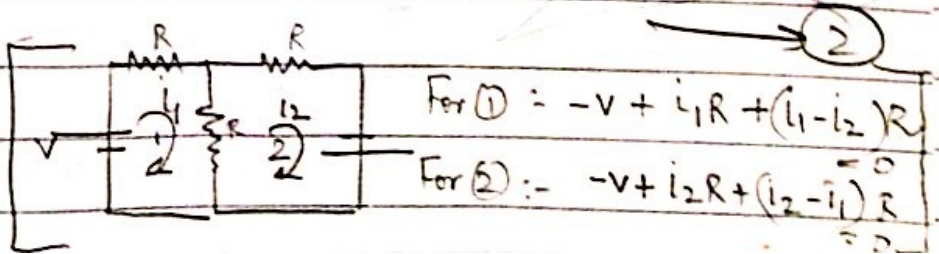
- node 1 (M_1)

$$F(t) = M_1 \frac{d^2x_1}{dt^2} + f_1 \frac{dx_1}{dt} + k_1 x_1 + f_{12} \frac{d}{dt}(x_1 - x_2) \quad \rightarrow (1)$$

[M_1 has influence of k_1, f_{12} & f_1]

- node 2 (M_2)

$$0 = M_2 \frac{d^2x_2}{dt^2} + f_2 \frac{dx_2}{dt} + k_2 x_2 + f_1 \frac{d}{dt}(x_2 - x_1) \quad \rightarrow (2)$$



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From (1), applying LT

$$F(s) = M_1 s^2 X_1(s) + f_1 s X_1(s) + k_1 X_1(s) + f_{12} s X_1(s) - f_{12} s X_2(s)$$

$\because$ Initial = 0.

$$\Rightarrow F(s) = (M_1 s^2 + f_1 s + k_1 + f_{12} s) X_1 s - (f_{12} s) X_2(s) \quad \text{--- (3)}$$

From (2), $0 = M_2 s^2 X_2(s) + f_2 s X_2(s) + k_2 X_2(s) + f_{12} s X_2(s) - f_{12} s X_1(s)$

$$0 = (M_2 s^2 + f_2 s + k_2 + f_{12} s) X_2(s) - (f_{12} s) X_1(s)$$

$$\Rightarrow X_1(s) = \frac{(M_2 s^2 + f_2 s + k_2 + f_{12} s) X_2(s)}{f_{12} s} \quad \text{--- (4)}$$

Using (4) in (3)

$$\Rightarrow F(s) = \frac{(M_1 s^2 + f_1 s + k_1 + f_{12} s)(M_2 s^2 + f_2 s + k_2 + f_{12} s) X_2(s)}{f_{12} s} - f_{12} s X_2(s)$$

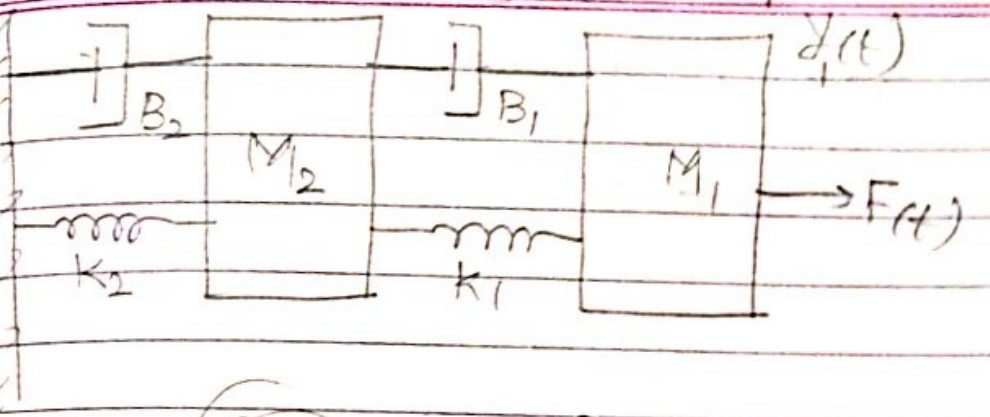
$$\Rightarrow F(s) = X_2(s) \left[\frac{(M_1 s^2 + f_1 s + k_1 + f_{12} s)(M_2 s^2 + f_2 s + k_2 + f_{12} s) - (f_{12} s)^2}{(f_{12})(s)} \right]$$

$$\Rightarrow \frac{X_2(s)}{F(s)} = \frac{f_{12} \cdot s}{(M_1 s^2 + f_1 s + k_1 + f_{12} s)(M_2 s^2 + f_2 s + k_2 + f_{12} s) - (f_{12} s)^2}$$

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* For a transfer fn, initial condns are 0. So, LT reduces to:-

$\mathcal{L}\left(\frac{d}{dt}x\right) = sX(s)$ & $\mathcal{L}\left(\frac{d^2x}{dt^2}\right) = s^2X(s)$



Find $G(s)$ from above system.

$$\rightarrow G(s) = \frac{Y_2(s)}{F(s)}$$

At node 1 (M_1)

$$F(t) = K_1(Y_1 - Y_2) + B_1\frac{d(y_1-y_2)}{dt} + M_1\frac{d^2y_1}{dt^2} \rightarrow (1)$$

At node 2 (M_2)

$$0 = B_2\frac{dy_2}{dt} + K_2y_2 + M_2\frac{d^2y_2}{dt^2} + B_1\frac{d}{dt}(y_2-y_1) + K_1(y_2-y_1) \rightarrow (2)$$

Applying LT on (1).

$$\mathcal{L}(F(t)) = \mathcal{L}\left(K_1(y_1-y_2) + B_1\frac{d}{dt}(y_1-y_2) + M_1\frac{d^2}{dt^2}y_1\right)$$

$$\Rightarrow F(s) = K_1(Y_1(s) - Y_2(s)) + B_1\mathcal{L}(Y_1(s) - Y_2(s)) + M_1(s^2Y_1(s))$$

$$\Rightarrow F(s) = Y_1(s)(K_1 + B_1s + M_1s^2) - Y_2(s)(K_1 + B_1s) \rightarrow (3)$$

Applying LT on (2), we get

$$\mathcal{L}(0) = \mathcal{L}\left(B_2\frac{dy_2}{dt} + K_2y_2 + M_2\frac{d^2y_2}{dt^2} + B_1\frac{d}{dt}(y_2-y_1) + K_1(y_2-y_1)\right)$$

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$$\Rightarrow 0 = B_2 s Y_2(s) + K_2 Y_2(s) + M_2 s^2 Y_2(s) + B_1 s(Y_2(s) - Y_1(s)) + K_1(Y_2(s) - Y_1(s))$$

$$\Rightarrow 0 = Y_2(s)(B_2 s + K_2 + s^2 M_2 + B_1 s + K_1) - Y_1(s)(sB_1 + K_1)$$

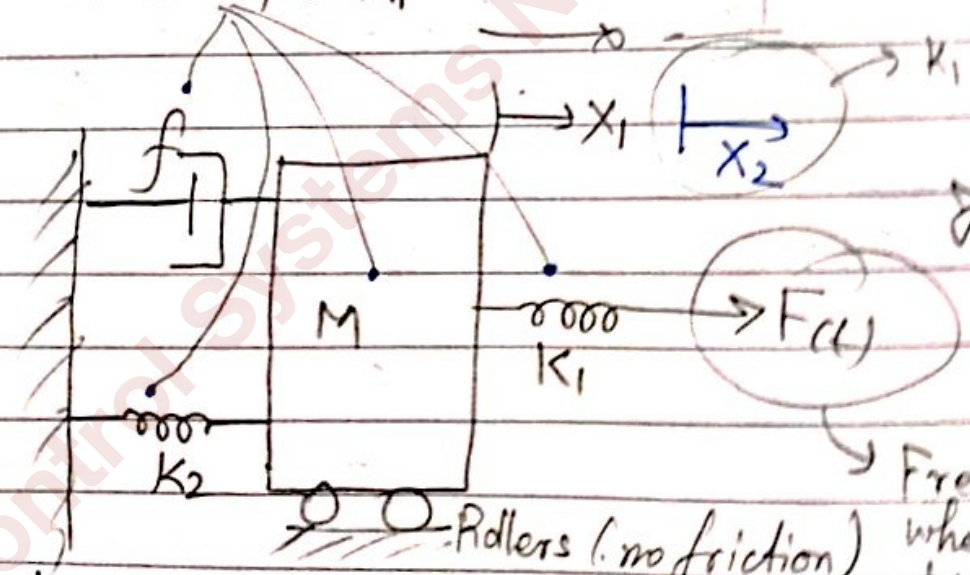
$$\Rightarrow Y_1(s) = Y_2(s)\left(\frac{K_1 + K_2 + sB_1 + B_2 s + s^2 M_2}{K_1 + sB_1}\right) \longrightarrow (4)$$

Using (4) in (3), we get

$$F(s) = \left[\left(\frac{K_1 + K_2 + sB_1 + B_2 s + s^2 M_2}{K_1 + sB_1}\right)(K_1 + B_1 s + M_1 s^2) - (K_1 + B_1 s)\right] Y_2(s)$$

$$\Rightarrow \frac{Y_2(s)}{F(s)} = \frac{(K_1 + B_1 s)^2}{(K_1 + K_2 + B_1 s + B_2 s + M_2 s^2)(K_1 + B_1 s + M_1 s^2) - (K_1 + B_1 s)}$$

4 elemts. for X_1



Find: $\frac{X_1(s)}{F(s)}$

Node 1

$$0 = K_1(x_1 - x_2) + f\frac{dx_1}{dt} + K_2 x_1 + M\frac{d^2x_1}{dt^2} \longrightarrow (1)$$

Node 2

$$F(t) = K_1(x_2 - x_1) \longrightarrow (2)$$

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Applying LT on (1)

$$\Rightarrow 0 = K_1(X_{1(s)} - X_{2(s)}) + fsX_{1(s)} + K_2X_1 + Ms^2X_{1(s)}$$

Applying LT on (2)

$$F(s) = K_1(X_{2(s)} - X_{1(s)}) \longrightarrow (4)$$

$$\Rightarrow X_2(s) = \left(\frac{K_1 + fs + K_2 + Ms^2}{K_1}\right) X_1(s) \longrightarrow (3)$$

Using (3) in (4)

$$\Rightarrow F(s) = \cancel{K_1}\left(\frac{\cancel{K_1} + fs + K_2 + Ms^2 - \cancel{K_1}}{\cancel{K_1}}\right) X_1(s)$$

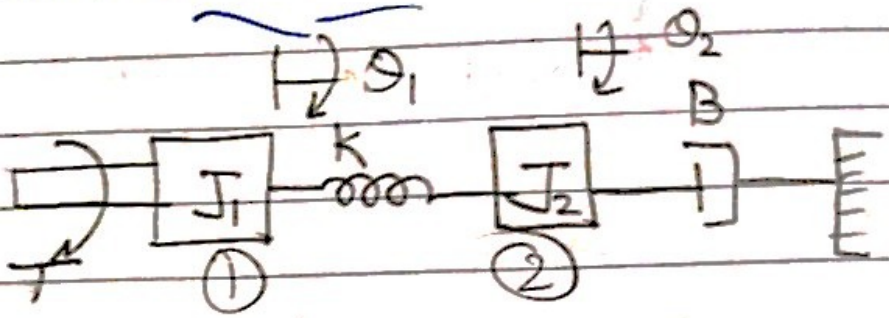
$$\Rightarrow \frac{X_1(s)}{F(s)} = \frac{1}{fs + K_2 + Ms^2}$$

Ans

Control Systems Notes - Akshansh

ROTATIONAL SYSTEM

Q.



Find $\frac{\theta_1(s)}{T(s)}$ =

At node (1)

$$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta_2) \rightarrow (1)$$

At node (2)

$$0 = J_2 \frac{d^2\theta_2}{dt^2} + K(\theta_2 - \theta_1) + B\frac{d\theta_2}{dt} \rightarrow (2)$$

Applying LT on eqn (1).

$$\Rightarrow T(s) = J_1 s^2 \theta_1(s) + K(\theta_1(s) - \theta_2(s)) \rightarrow (3)$$

LT on eqn (2)

$$\Rightarrow 0 = J_2 s^2 \theta_2(s) + K(\theta_2(s) - \theta_1(s)) + Bs\theta_2(s)$$

$$\Rightarrow \theta_1(s) = \left(\frac{J_2 s^2 + K + Bs}{K}\right)\theta_2(s) \rightarrow (5)$$

From (3)

$$T(s) = \theta_1(s)\left[J_1 s^2 + K\right] - K\left(\frac{K}{J_2 s^2 + K + Bs}\right)\theta_1(s)$$

$$\Rightarrow T(s) = \theta_1(s)\left[\frac{(J_2 s^2 + K + Bs)(J_1 s^2 + K) - K^2}{J_2 s^2 + K + Bs}\right]$$

Control Systems Notes - Akshansh

$$\Rightarrow \frac{\Theta_1(s)}{T(s)} = \frac{J_2 s^2 + K + Bs}{(J_2 s^2 + K + Bs)(J_1 s^2 + k) - K^2}$$

Ans

✱ GEAR TRAIN SYSTEM (analogous to Transformer)

→ ✓ used for transforming torque & hence speed

- mechanical sys.
- electrical sys.

- Notations :-

N_1, N_2 : no. of teeth in gear 1 & gear 2 resp.

r_1, r_2 : radius of gear 1 & gear 2 resp.

θ_1, θ_2 : Analogous display of motor shaft, load shaft resp.

J_1 : moment of inertia of motor & gear 1

J_2 : " " " gear 2 & load

B_1 : friction of motor & gear 1

B_2 : " gear 2 & load

* Refer to derivⁿ on gear train sys.

T_L : load torque

T_2 : Torque transmitted to gear 2

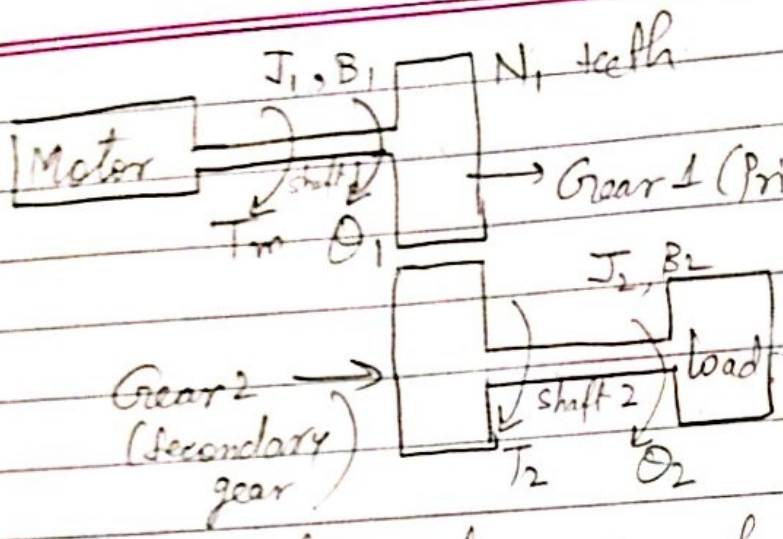
T_m : " developed by motor

T_1 : load torque on gear 1

ω_1, ω_2 : avg. speed of motor shaft & load shaft resp.

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Torque developed motor = Sum of load torque req^d on gear 1 & opposing torque of J_1 & B_1

$$T_m = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + T_1 \longrightarrow (1)$$

$$T_2 = J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + T_2 \longrightarrow (2)$$

When motor drives the load, linear distance travelled along surface of each gear is same

$$\theta_1 r_1 = \theta_2 r_2$$

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} \longrightarrow (3)$$

Now, no. of teeth in each gear $\propto$ its radius

$$\Rightarrow N_1 \propto r_1 \quad \& \quad N_2 \propto r_2$$

$$\text{or} \quad \frac{N_1}{N_2} = \frac{r_1}{r_2} \longrightarrow (4)$$

From (3) & (4)

$$\frac{\theta_2}{\theta_1} = \frac{N_2}{N_1} \longrightarrow (5)$$

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Ideal gear train is $\rightarrow$ no power loss in transmission
$\therefore$ work done by both gear are equal

$$WS = T_1\theta_1 = T_2\theta_2 \longrightarrow (6)$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} \longrightarrow (7)$$

differentiating (6) w.r.t. t, we get

$$T_1 \frac{d\theta_1}{dt} = T_2 \frac{d\theta_2}{dt} \longrightarrow (8)$$

$$\Rightarrow T_1\omega_1 = T_2\omega_2$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{\omega_2}{\omega_1} \longrightarrow (9)$$

differentiating (8) w.r.t. t

$$\rightarrow T_1 \frac{d^2\theta_1}{dt^2} = T_2 \frac{d^2\theta_2}{dt^2}$$

$$\Rightarrow T_1\alpha_1 = T_2\alpha_2$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{\alpha_2}{\alpha_1} \longrightarrow (10)$$

α_1, α_2: angular accel^n of motor & load shaft resp

Now, from (3), (4), (7), (9), (10),

$$\frac{N_1}{N_2} = \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} = \frac{T_1}{T_2} = \frac{\omega_2}{\omega_1} = \frac{\alpha_2}{\alpha_1} \longrightarrow (11)$$

Note: All the 3 mechanical parameters: θ, ω, α are inverse ratio.

When $N_1 > N_2$: gear train ↑, speed ↓, Torque ↑.

$N_1 = N_2$: no change in speed & torque.

$N_1 < N_2$: speed ↓, torque ↑.

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Deriving torque eq" referred to shaft ①

From eq" ⑪, $\alpha_2 = \alpha_1 \frac{N_1}{N_2}$.

$$\Rightarrow \frac{d^2\theta_2}{dt^2} = \frac{d^2\theta_1}{dt^2}\frac{N_1}{N_2} \longrightarrow (12)$$

From eq" ⑪, $\omega_2 = \omega_1 \frac{N_1}{N_2}$.

$$\& \quad \frac{d\theta_2}{dt} = \frac{d\theta_1}{dt}\frac{N_1}{N_2} \longrightarrow (13)$$

From eq" ⑪ $T_2 = T_1 \frac{N_2}{N_1} \longrightarrow (14)$.

Substituting above in eqn ②.

$$J_2 \frac{d^2\theta_1}{dt^2}\frac{N_1}{N_2} + B_2 \frac{d\theta_1}{dt}\frac{N_1}{N_2} + T_L = T_1 \frac{N_2}{N_1}$$

$$\times \left(\frac{N_1}{N_2}\right) \Rightarrow J_2 \frac{d^2\theta_1}{dt^2}\left(\frac{N_1}{N_2}\right)^2 + B_2 \frac{d\theta_1}{dt}\left(\frac{N_1}{N_2}\right)^2 + T_L\left(\frac{N_1}{N_2}\right) = T_1 \longrightarrow (15)$$

Substituting above in ①.

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + J_2 \frac{d^2\theta_1}{dt^2}\left(\frac{N_1}{N_2}\right)^2 + B_2 \frac{d\theta_1}{dt}\left(\frac{N_1}{N_2}\right)^2 + T_L\left(\frac{N_1}{N_2}\right) = T_m.$$

☆

$$\Rightarrow T_m = \underbrace{\left[J_1 + J_2\left(\frac{N_1}{N_2}\right)^2\right]}_{J_{eq}\,1}\frac{d^2\theta_1}{dt^2} + \underbrace{\left[B_1 + B_2\left(\frac{N_1}{N_2}\right)^2\right]}_{B_{eq}\,①}\frac{d\theta_1}{dt} + T_L\left(\frac{N_1}{N_2}\right) \longrightarrow (16)$$

Control Systems Notes AKshansh

moment of inertia.

here,

J_{eq1} : equivalent MOI referred to motor shaft.

B_{eq1} : equivalent friction coeff. referred to shaft ①

eqn (6) is called torque eqn referred to motor shaft.

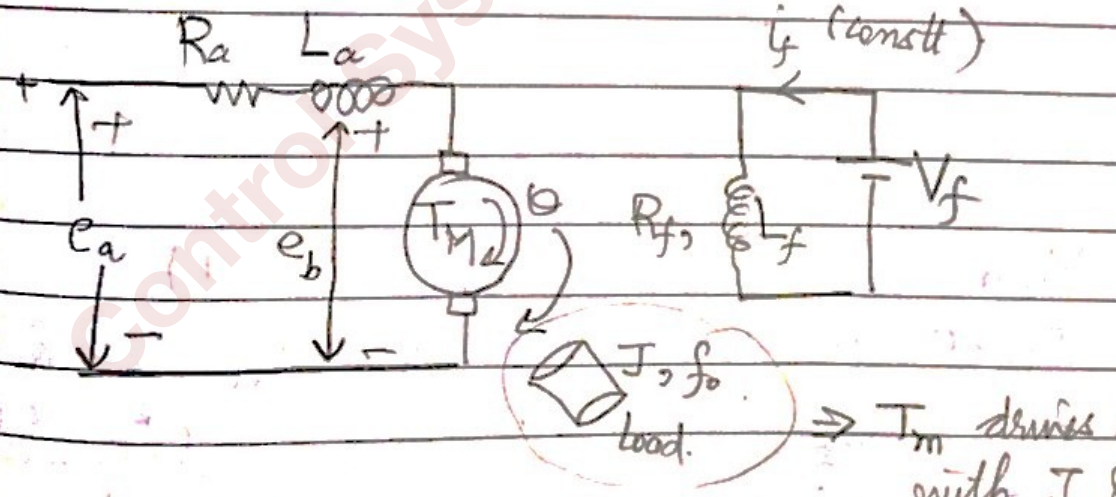
Similarly, do torque eqn refered to load shaft.

$$J_{eq2}\ddot{\theta}_2 + f_{eq2}\dot{\theta}_2 + T_L = \left(\frac{N_2}{N_1}\right) T_m$$

→ do derivn of this also

§ DC Motor Armature Control

→ method used when we want to get speed above rated speed.



⇒ T_m drives a load with J & f_o.

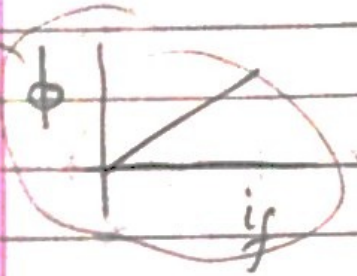
Units :- • i_a, i_f : A
• e_a, e_b : V
• T_m : N-m
• θ : rad

Control Systems Notes - Ayushansh

- $J_{equivalent}$ MOI of motor & load referred to motor shaft : kg-m²
- f_0 & Viscous friction coeff. of motor & load $_{equivalent}$ referred to motor shaft : N-m/rad/s

DC servo motor –
- high starting torque
- low inertia
- linear range of magnetiz^n curve.

assuming linear magnetiz^n curve



$\phi \propto i_f$, $\phi = K_f i_f$; $T_M \propto \phi i_a$
$= K_1 K_f i_f i_a$

For armature control, or $T_M = K_T i_a$ → (1)
i_f is constt.

(In DC series motor, done last sem, $i_f = i_a$
So, $T_M = K_T i_a^2$ (2)
So, it can drive larger load)

Now
$e_b \propto$ speed $= K_b \frac{d\theta}{dt}$; $e_a = L_a \frac{di_a}{dt} + R_a i_a + e_b$
e_b → back emf.
→ (2)
KVL → (3)

Now,
$J \frac{d^2\theta}{dt^2} + f_0 \frac{d\theta}{dt} = T_M = K_T i_a$ → (4)
→ eq^n of motor torque (done in prev. deriv^n)

Control Systems Notes - Akshansh

From ② & ④, $E_b(s) = K_b s \Theta(s)$ → ⑤

(Applying LT on ②)

$\Rightarrow (L_a s + R_a) I_a(s) = E_a(s) - E_b(s)$ → ⑥

(LT of ③)

&

$(Js^2 + f_0 s)\Theta(s) = T_M(s) = K_T I_a(s)$ → ⑦

(LT of ④)

Substituting Using ④ to ⑥, substituting ⑤ in ⑥

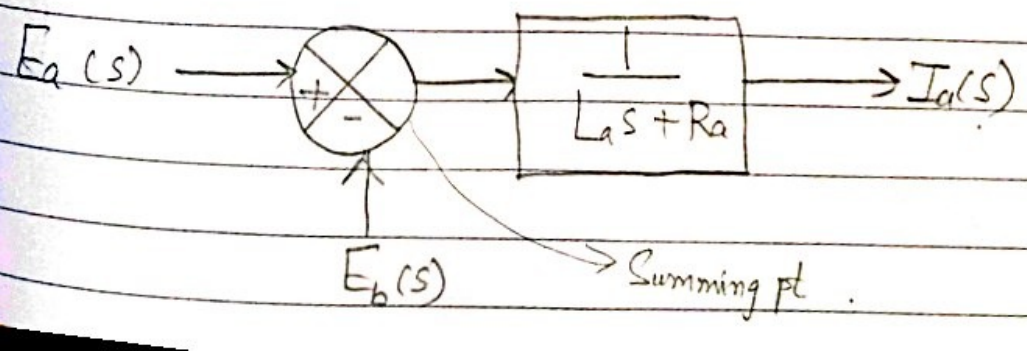
$$I_a(s) = \frac{E_a(s) - K_b s \Theta(s)}{L_a s + R_a} \rightarrow ⑧$$

Using ⑧ in ⑦.

$\Rightarrow (Js^2 + f_0 s)\Theta(s)[L_a s + R_a] = K_T E_a(s) - K_T K_b s \Theta(s)$

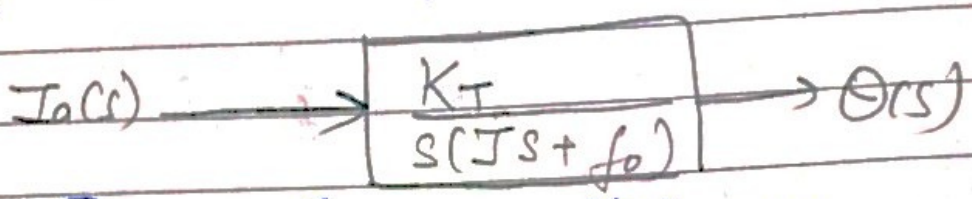
Now, $G(s) = \frac{\Theta(s)}{E_a(s)} = \frac{K_T}{(Js^2 + f_0 s)(L_a s + R_a) + K_T K_b s}$ → ⑨

Now, eqⁿ ⑥ is represented as:-

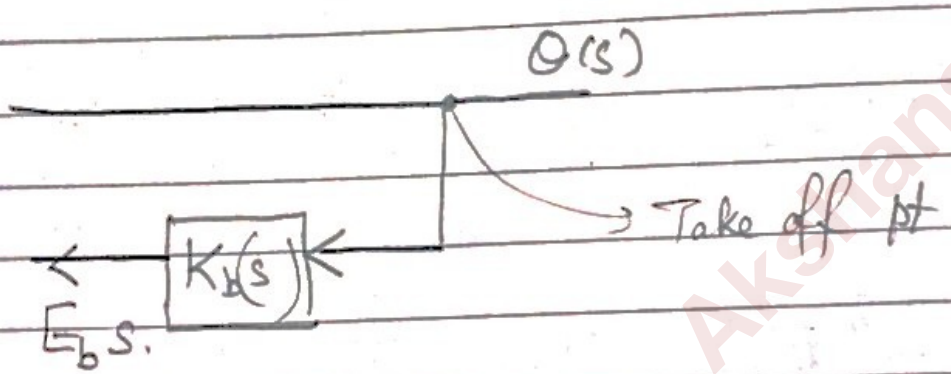


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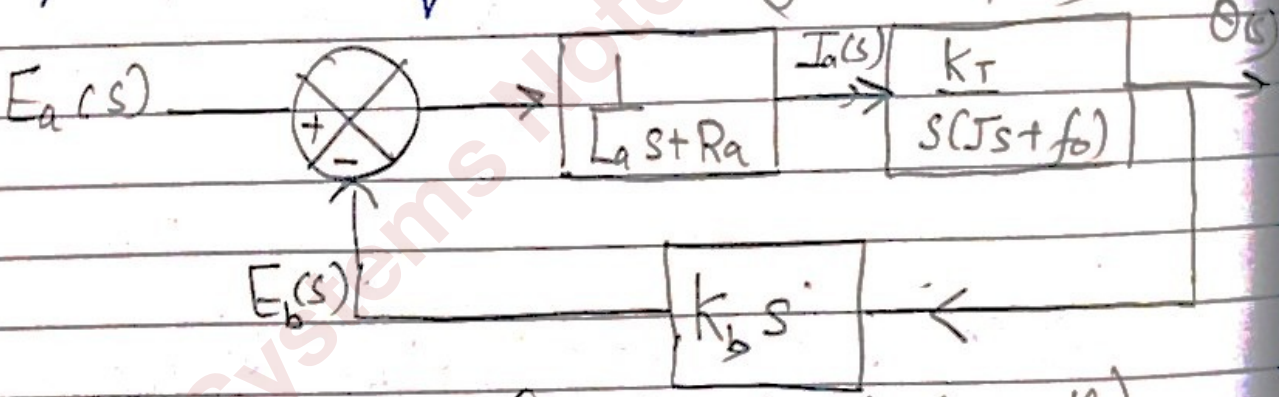
eqn (7) can be represented as :-



eqn (5) can be represented as :-

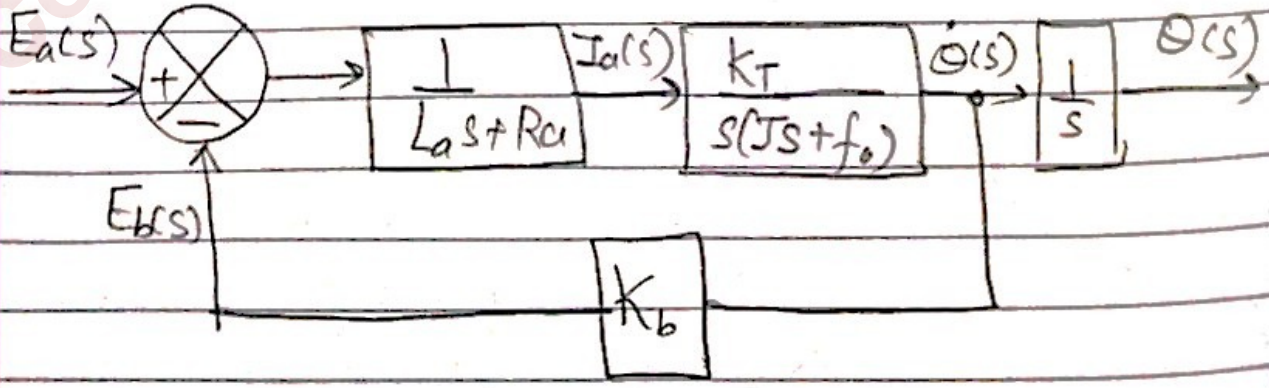


Complete block diagram :- (forward path)



(Negative feedback path)

Shifting the take off point.



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We know,

La is negligible in DC motor.

So, TF $= G(s) = \frac{\theta(s)}{E_a(s)} = \frac{K_T/R_a}{Js^2 + s(f_0 + K_T K_b/R_a)}$

As $f \propto \frac{d\theta}{dt}$ & $E_b \propto$ speed $\propto \frac{d\theta}{dt}$.

So, $f \propto E_b$.

Hence, Back emf effectively increases viscous friction.

Effective viscous friction coeff., $f = f_0 + \frac{K_T K_b}{R_a}$

$$TF = \frac{\theta(s)}{E_a(s)} = \frac{K_T/R_a}{s(Js+f)} = \frac{K_T/R_a}{fs\left(\frac{Js}{f}+1\right)}$$

$$= \frac{K_T/R_a f}{s\left(s\frac{J}{f}+1\right)}$$

$$\Rightarrow TF = \frac{K_m}{s(sT_m+1)}$$

- K_m: motor gain constt.
- $T_m = \frac{J}{f}$ = motor time constt.

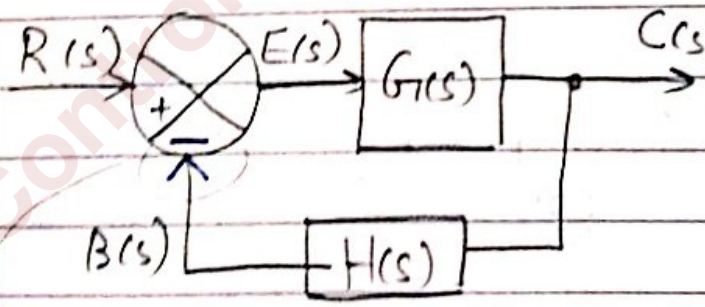
Control Systems Notes - Akshansh

Now,
- electrical power converted to mechanical form
  $= e_b i_a = K_b \dot{\theta} i_a$ W.
- Power at shaft (in mech. form) $= T\dot{\theta} = K_T i_a \dot{\theta}$
- Power balance eqn, $K_b \dot{\theta} i_a = K_T i_a \dot{\theta}$.
- $K_b = K_T$ (K_b can be measured more easily than K_T)

Self * Also do :- Derivation of DC motor field control.

BLOCK DIAGRAM ALGEBRA

✓ Block diagram of a closed loop sys. :-



$C(s) = G(s) \cdot E(s)$

$E(s) = R(s) - B(s)$

$H(s) = \frac{B(s)}{C(s)}$

f/b signal subtracted from i/p :) -ve f/b.

$E(s) = R(s) - H(s) \cdot C(s)$

$C(s) = G(s) \cdot E(s) = G(s)[R(s) - H(s)C(s)]$

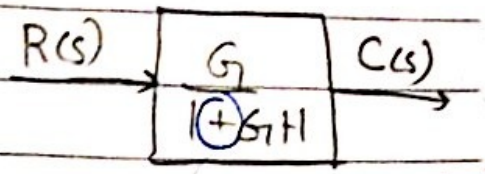
$\Rightarrow C(s) = G(s)R(s) - G(s)H(s)C(s)$

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called as Field → forward TF

Now,

$$\frac{C(s)}{R(s)} = T(s) = \frac{G(s)}{1+G(s)H(s)}$$



(for -ve feedback)

overall TF of overall loop (↙)

TF only for field forward path

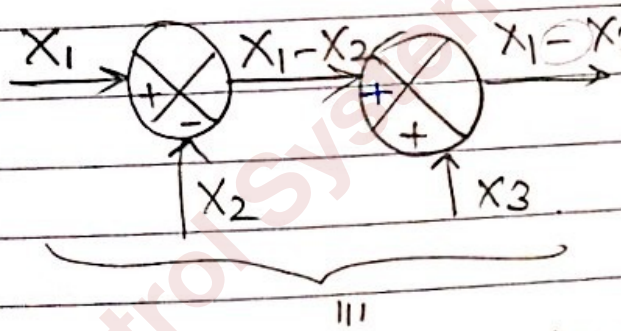
Feedback path TF

$$G(s) = \frac{C(s)}{E(s)}$$

+ve sign for -ve feedback (converse also true)

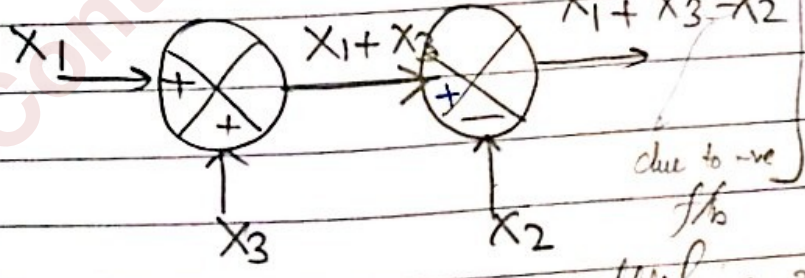
✱ Block Diagram Reduction

- **Rule 1 : Interchanging to summing points that are directly connected to each other.**



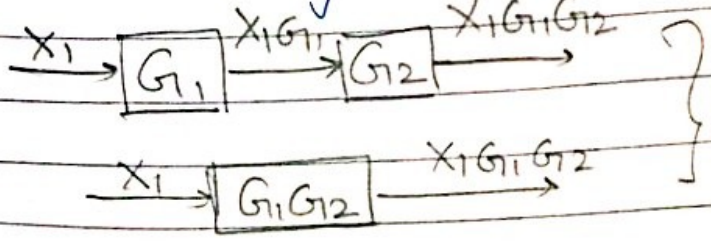
equivalent forms

[If some other elements are in b/w, o/p is not same]



due to -ve f/b

- **Rule 2 : Cascading** → multiplying 2 blocks.

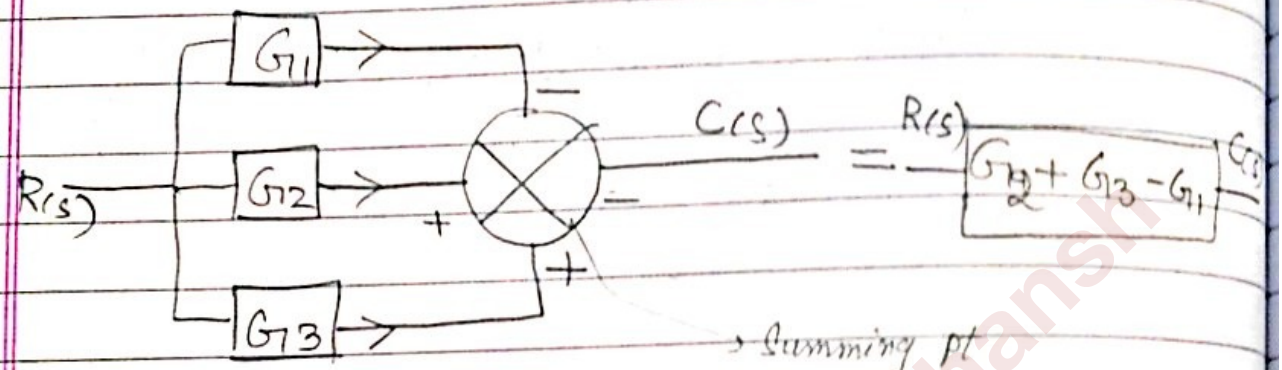


equivalent

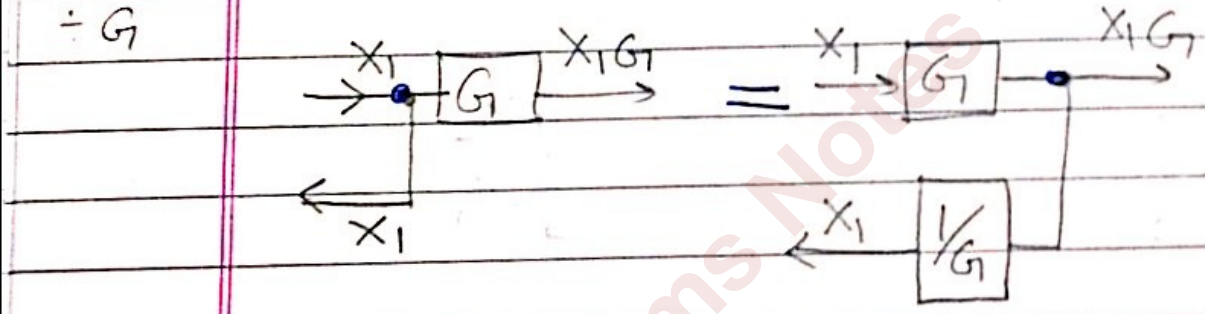
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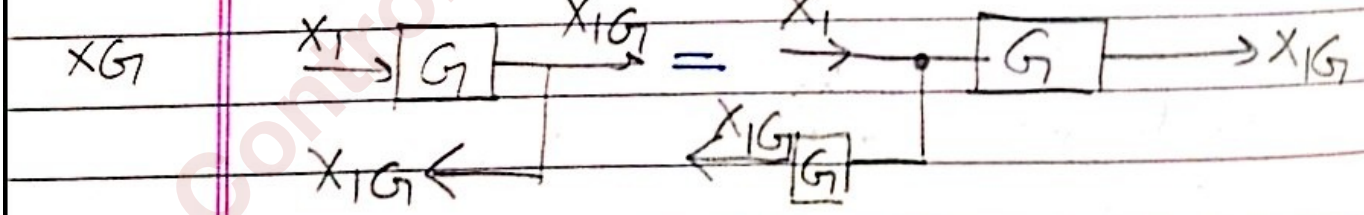
- Rule 3 : Blocks in Parallel (BIP)
  (Combining feed forward paths)



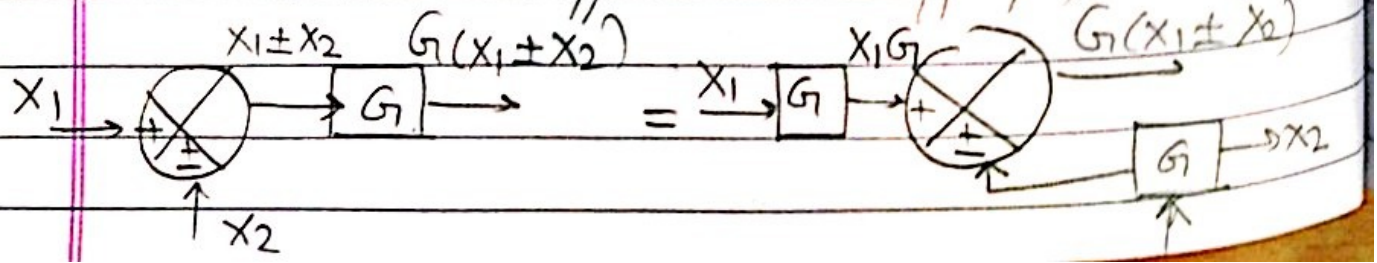
- Rule 4: moving a take off pt. after a block.



- Rule 5: Moving a take off pt before a block (i.e, ahead of a block)



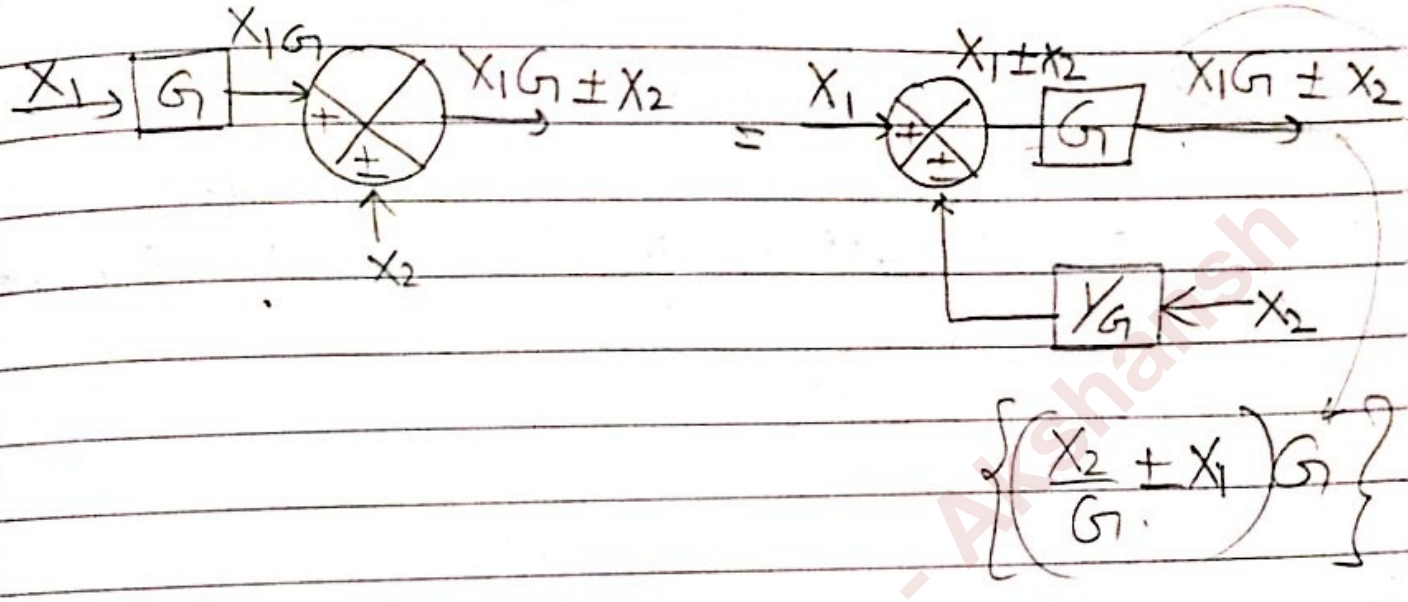
- Rule 6: Moving a summing pt. after a block (behaves opp. to take off pt.)



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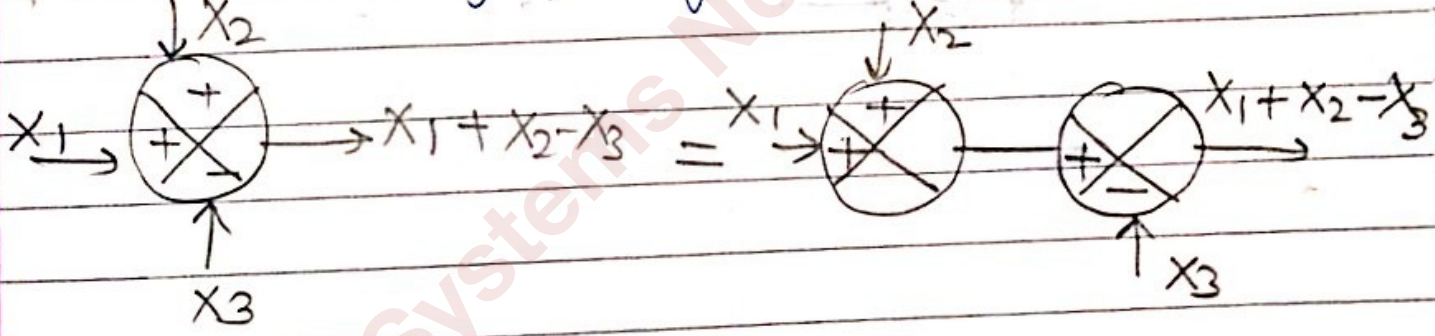
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- **Rule 7: Moving a summing pt ahead of a block (i.e, before the block)**

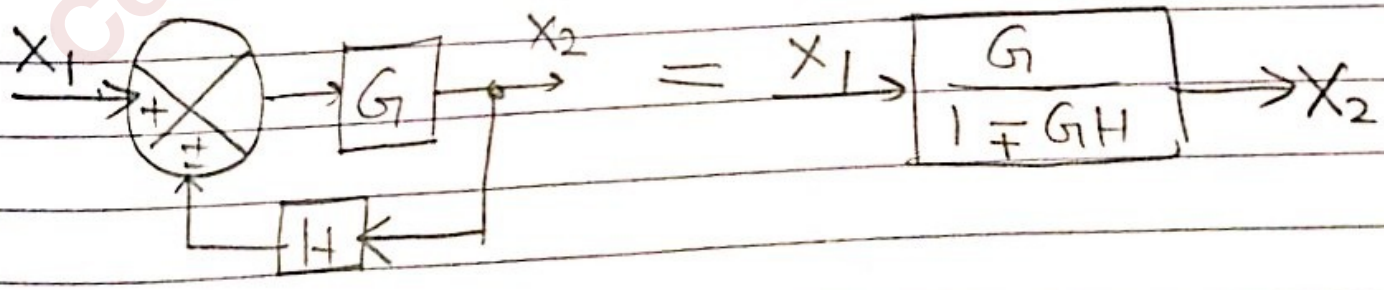


$$\left\{\left(\frac{X_2}{G_1} \pm X_1\right) G_1\right\}$$

- **Rule 8: Combining/Splitting a summing pt.**



- **Rule 9: Eliminating a feedback loop**
  → Minor feedback loop rule (MFL)



Control Systems Notes - Akshansh

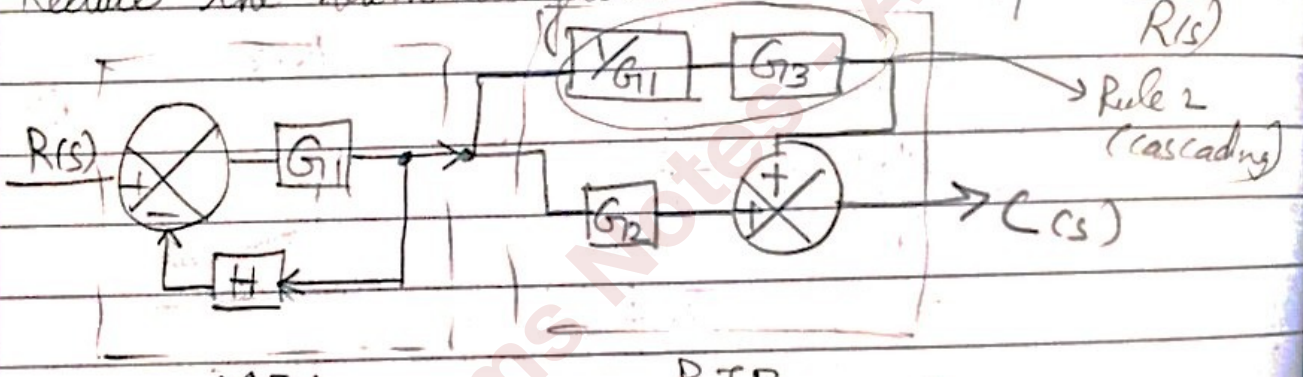
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→ General rules

☆ Procedure for block diagram reduction.

1. Reduce blocks in cascade.
2. Reduce blocks in parallel (feed forward) (BIP)
3. Reduce minor feedback loops (MFL)
4. As far as possible, try to shift take off points toward right and summing points toward left.

ex Reduce the block diagram shown to find $\frac{C(s)}{R(s)} = TF$



$= R(s) \rightarrow \boxed{G_1} \rightarrow \boxed{G_3/G_1 + G_2} \rightarrow C(s)$ (Rule 3)

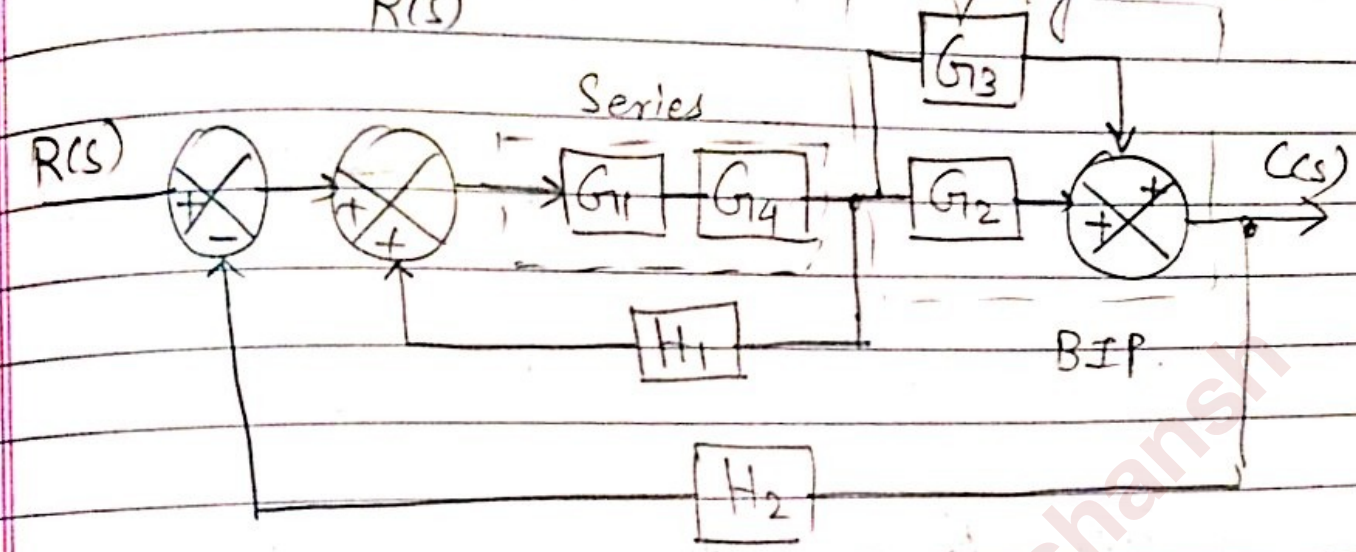
$= R(s) \rightarrow \boxed{\frac{G_1}{1+HG_1}} \rightarrow \boxed{G_3/G_1 + G_2} \rightarrow C(s)$ (Rule 9)

$= R(s) \rightarrow \boxed{1 \left(\frac{G_3 + G_1G_2}{G_1}\right)\left(\frac{G_1}{1+HG_1}\right)} \rightarrow C(s)$ (Rule 2) (Cascading)

$\Rightarrow TF = \frac{G_3 + G_1G_2}{1 + HG_1}$ Ans

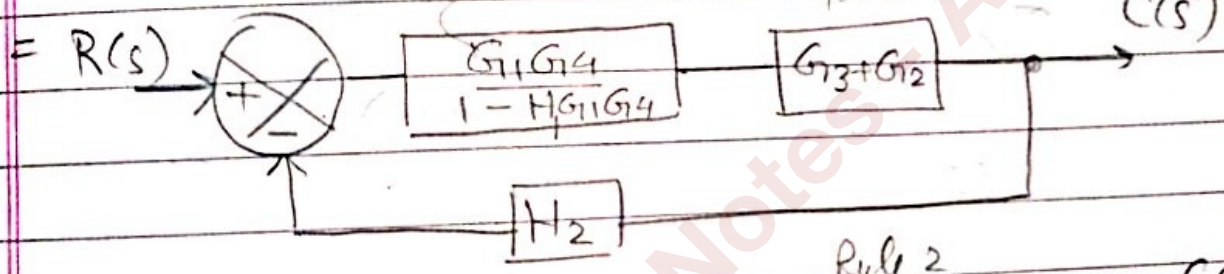
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Q. Determine $\frac{C(s)}{R(s)}$ for the following sys :-

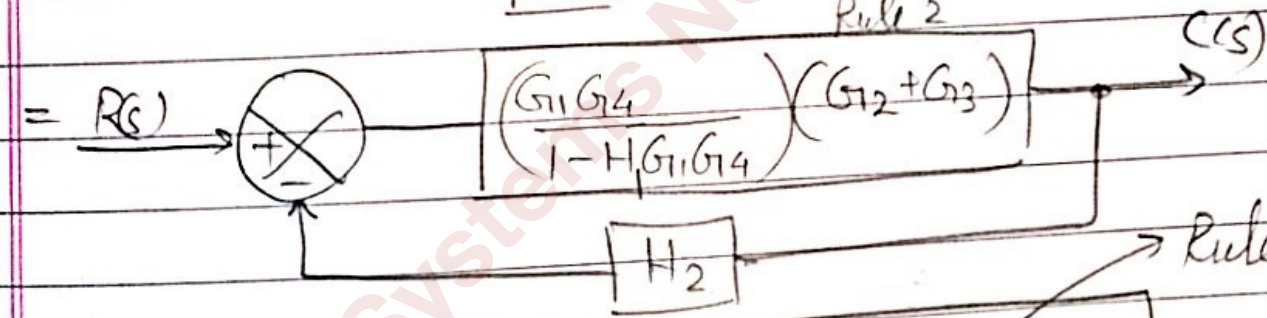


→ Rule 2 & 1

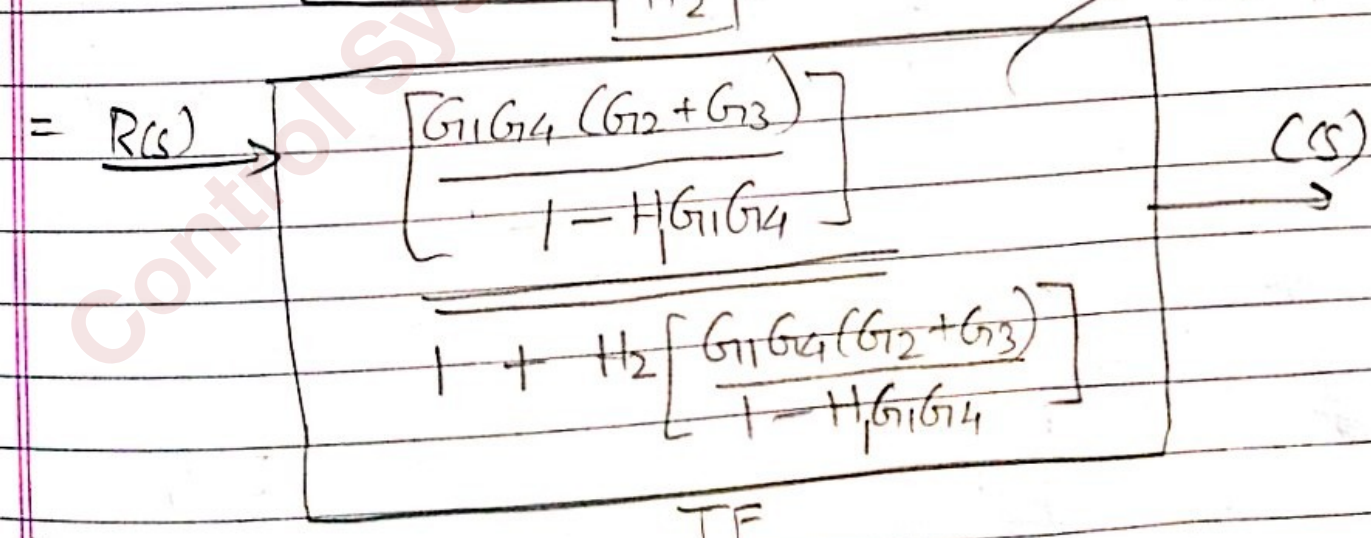
Rule 3



Rule 2



Rule 9

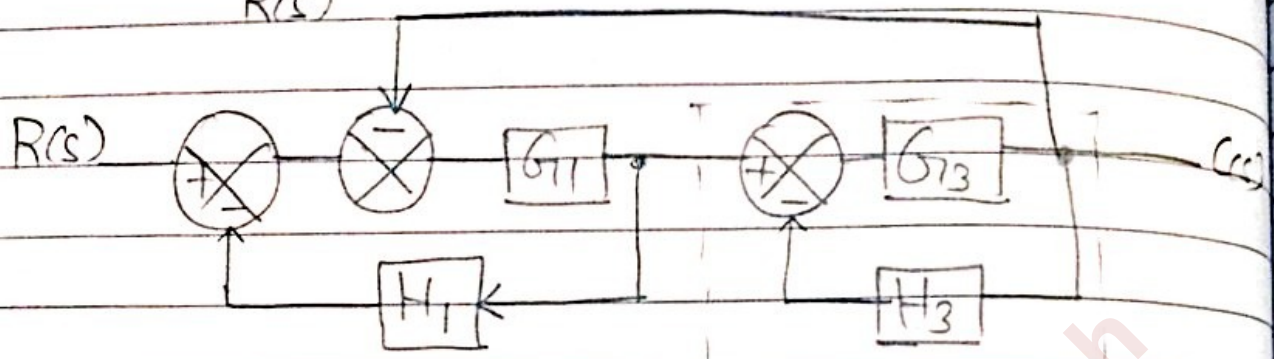


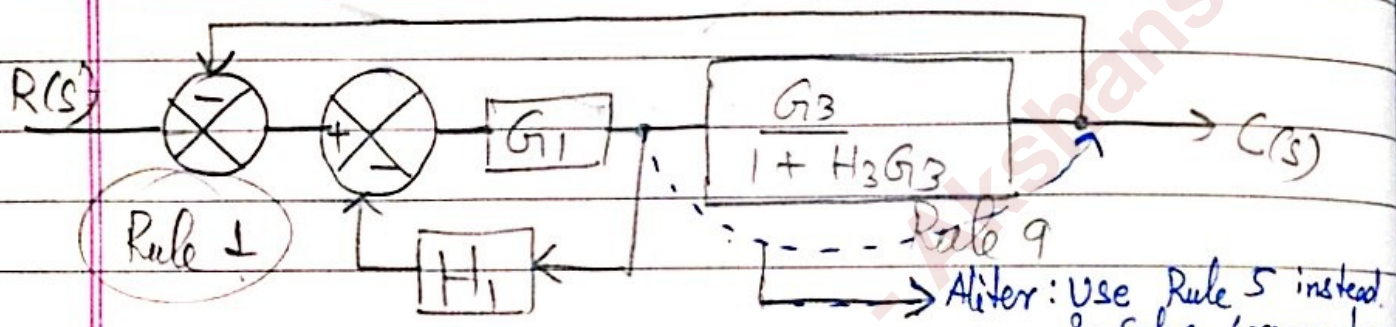
$$\Rightarrow TF = \frac{G_1 G_4 (G_2 + G_3)}{1 - H_1 G_1 G_4 + H_2 (G_1 G_4)(G_2 + G_3)}$$

Ans

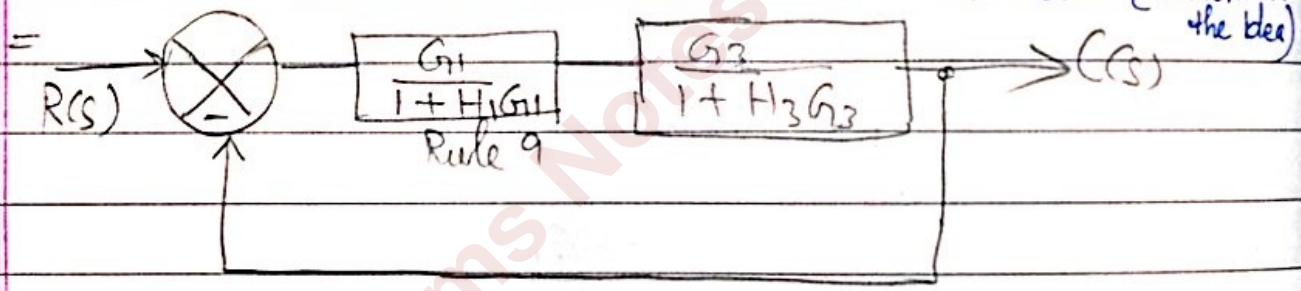
Control Systems Notes - Akshansh

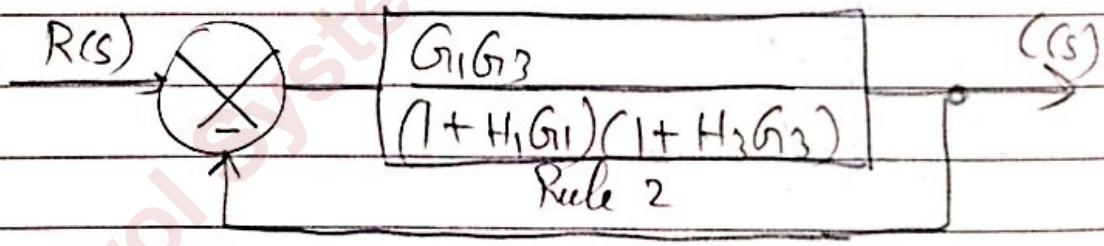
8. Obtain $\frac{C(s)}{R(s)}$

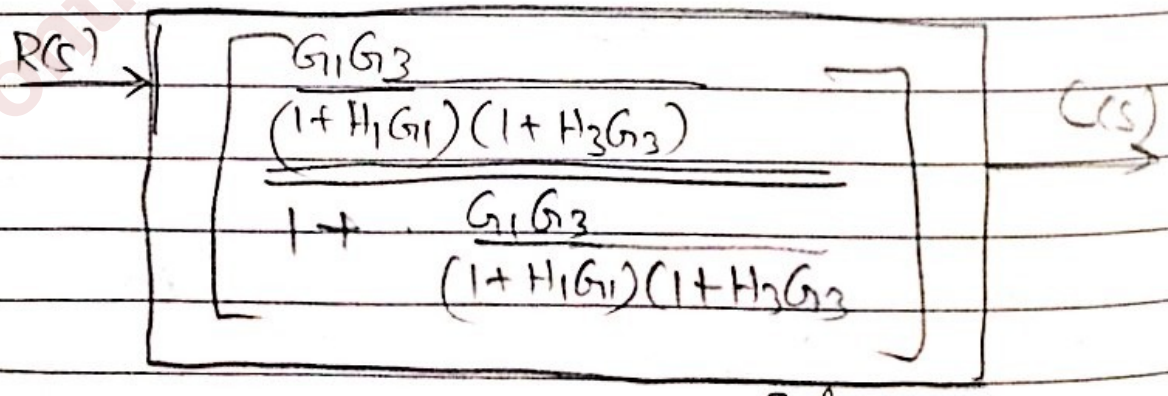




Alter: Use Rule 5 instead & solve (remember the idea)



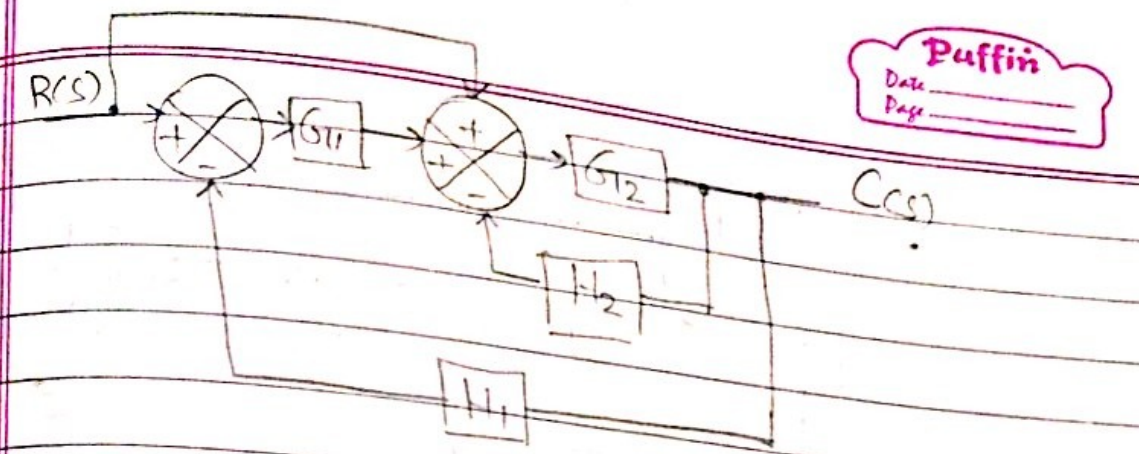




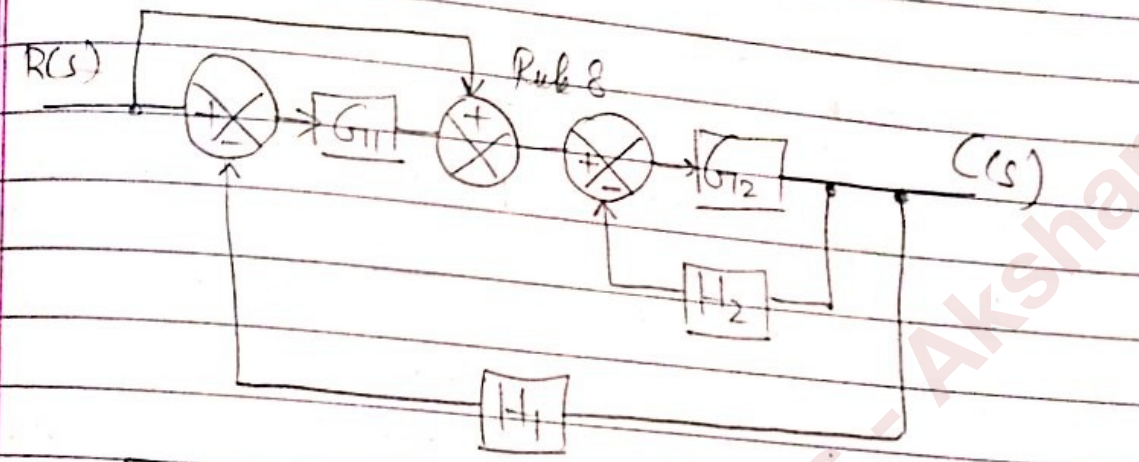
$$\Rightarrow TF = \frac{G_1 G_3}{(1 + H_1 G_1)(1 + H_3 G_3) + G_1 G_3}$$

Control Systems Notes - Akshansh

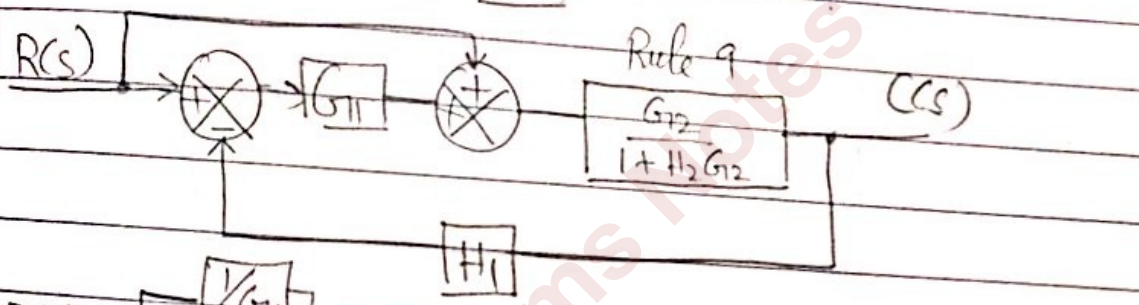
Q.



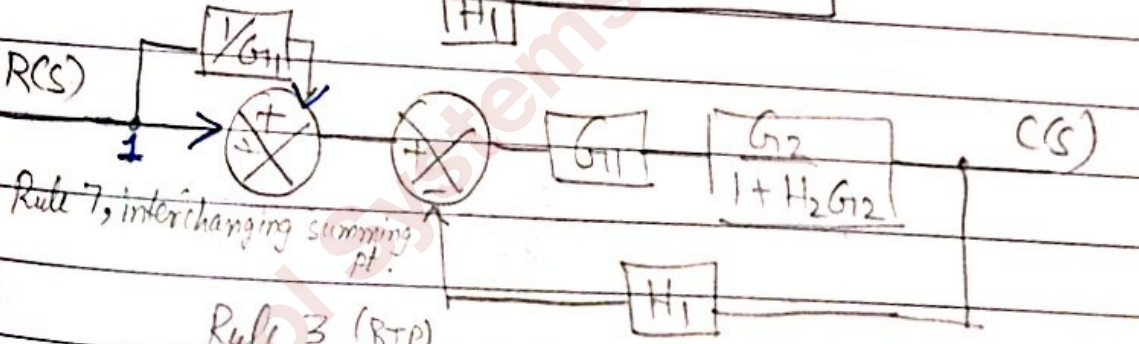
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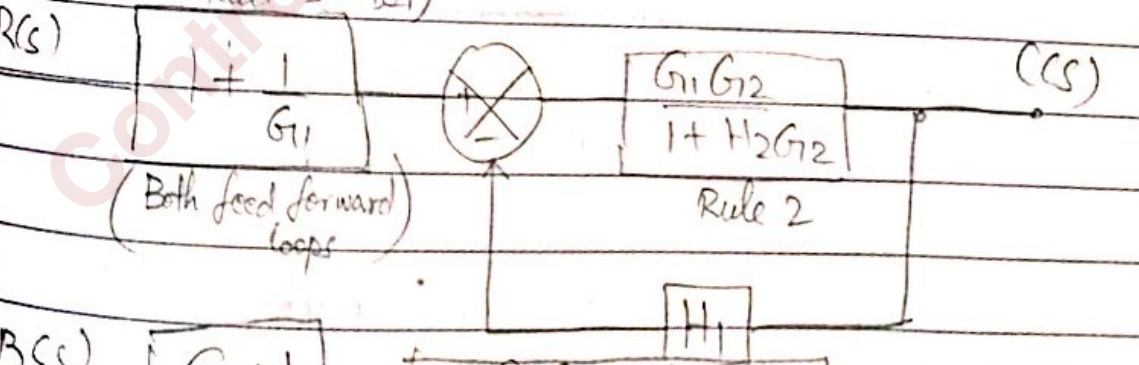
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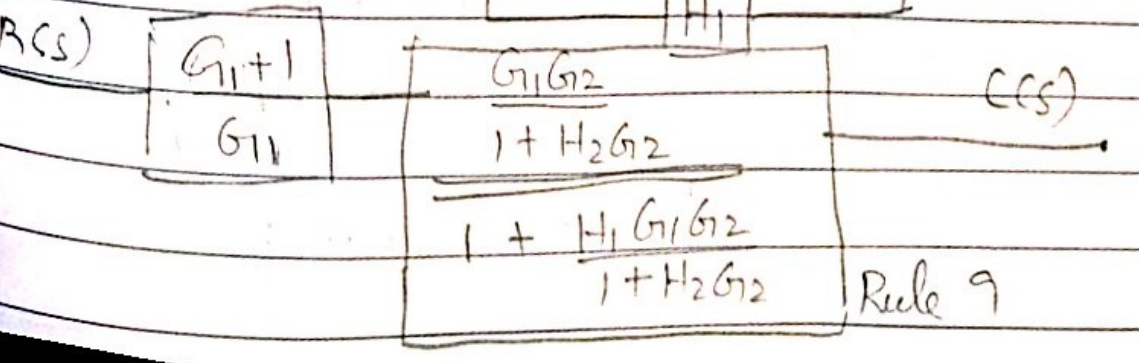
Rule 7, interchanging summing pt.

Rule 3 (BIP)

=



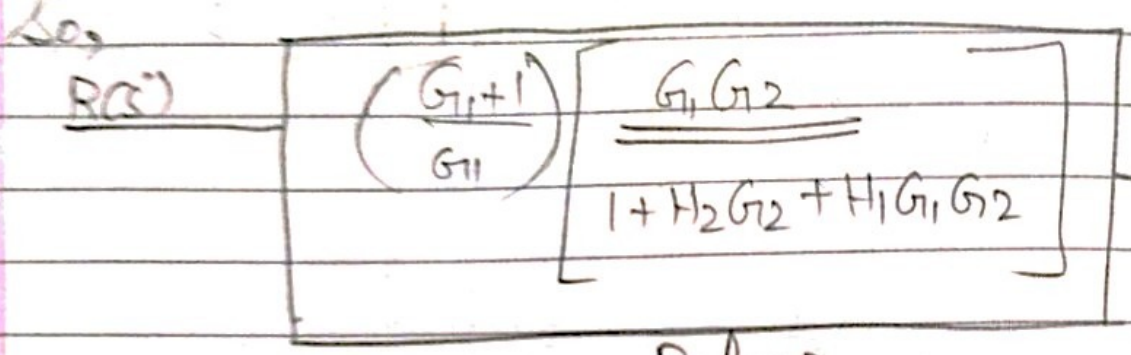
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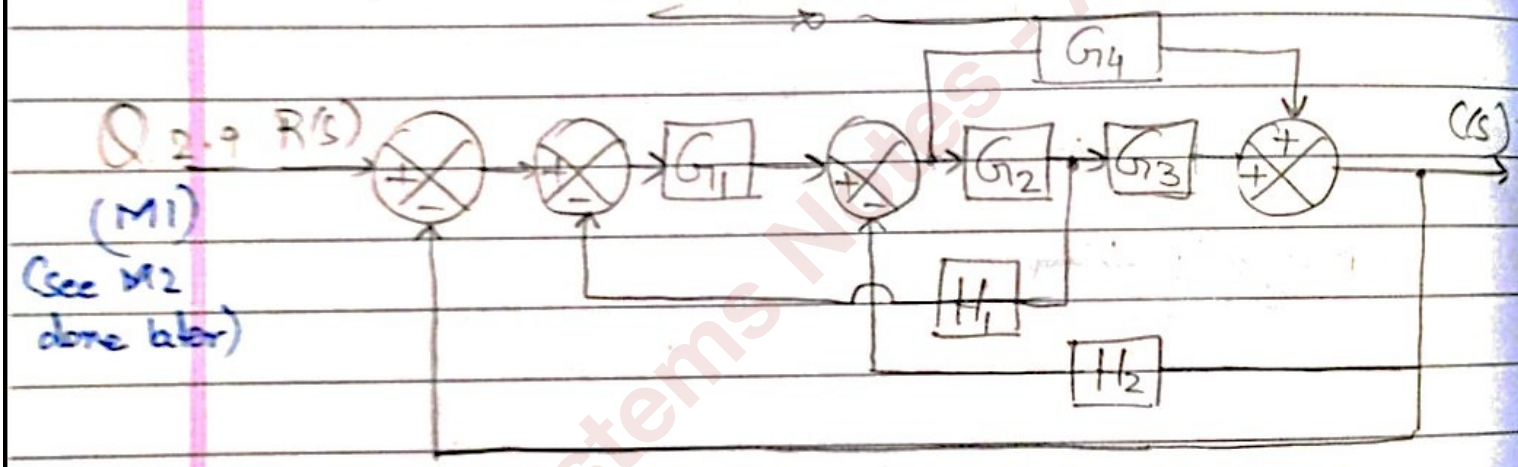
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So,

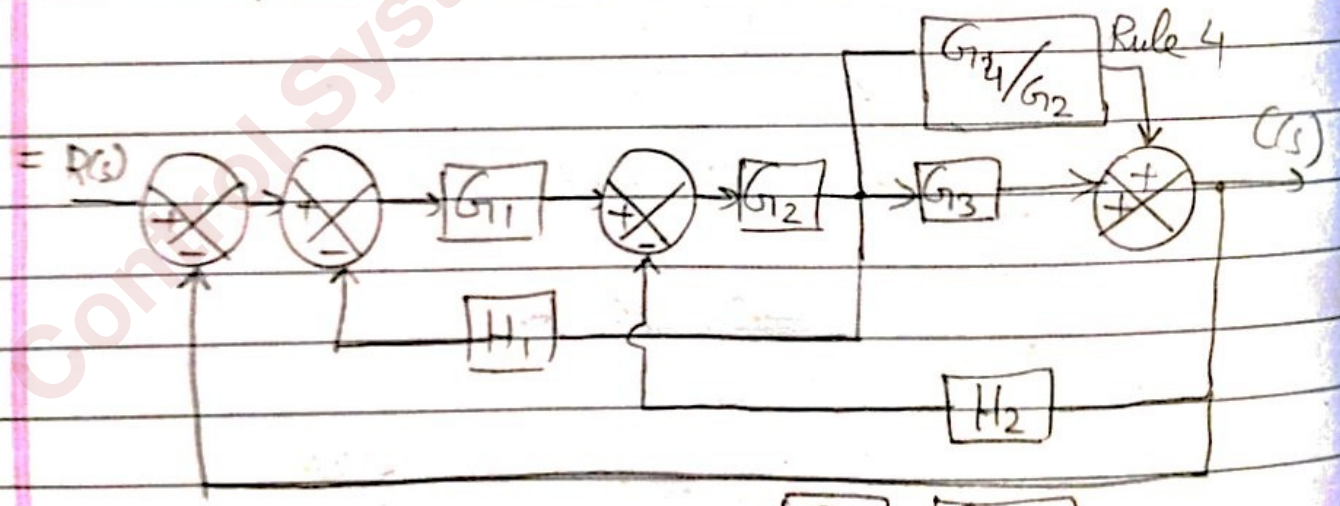


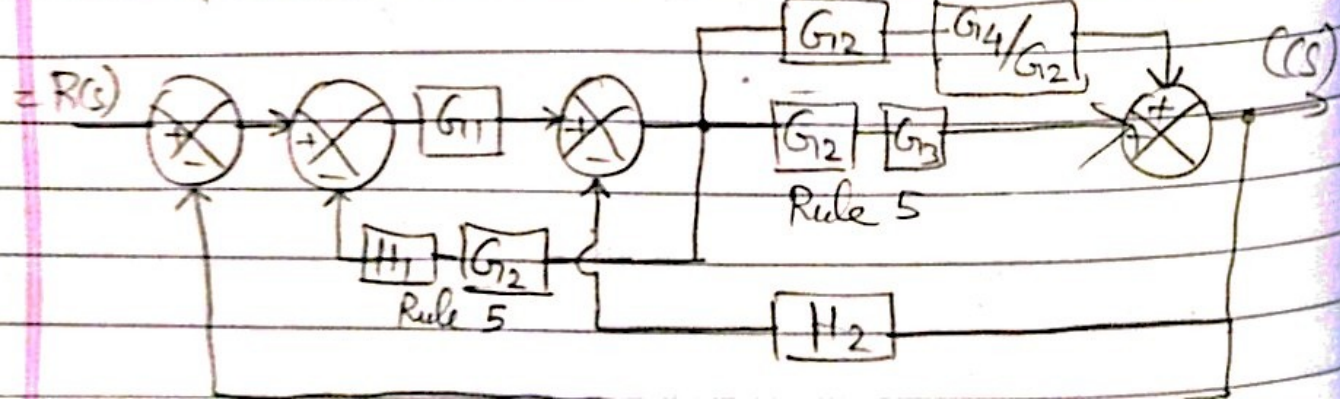
Rule 2

$$\Rightarrow \quad TF = \frac{C(s)}{R(s)} = \frac{(G_1+1)G_2}{(1+H_2G_2+H_1G_1G_2)}$$



(Q 2.9 (M1) (See M2 done later)

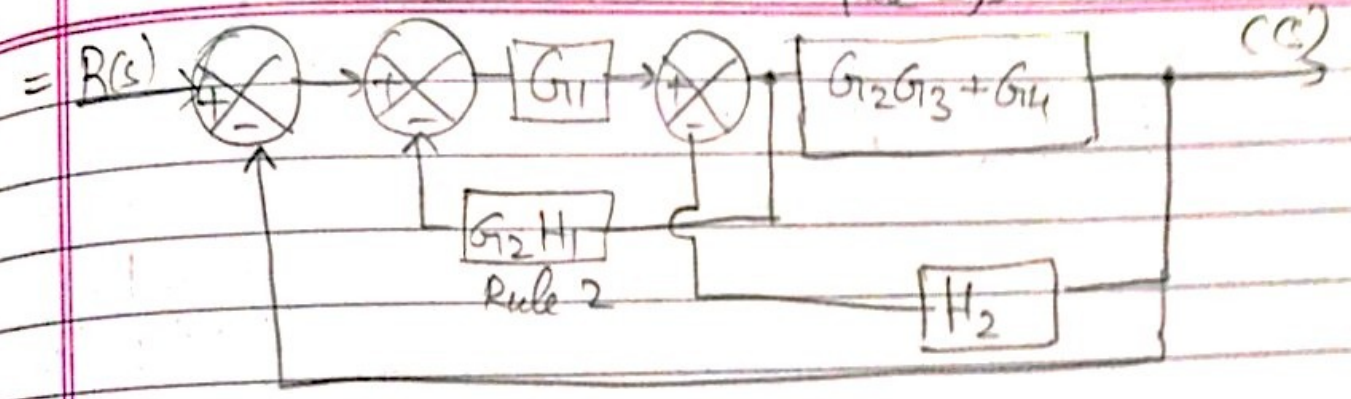


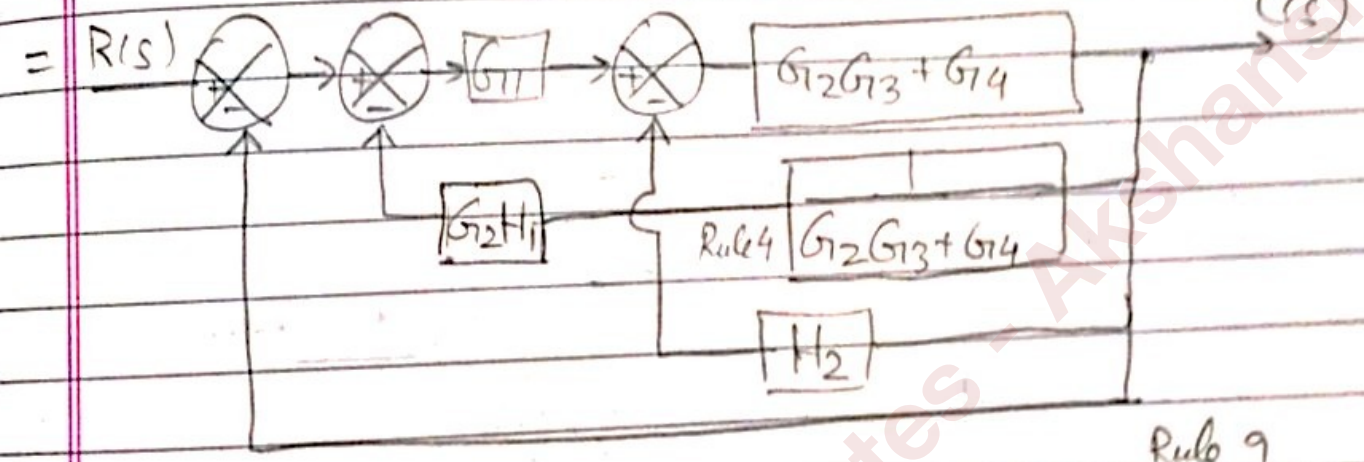


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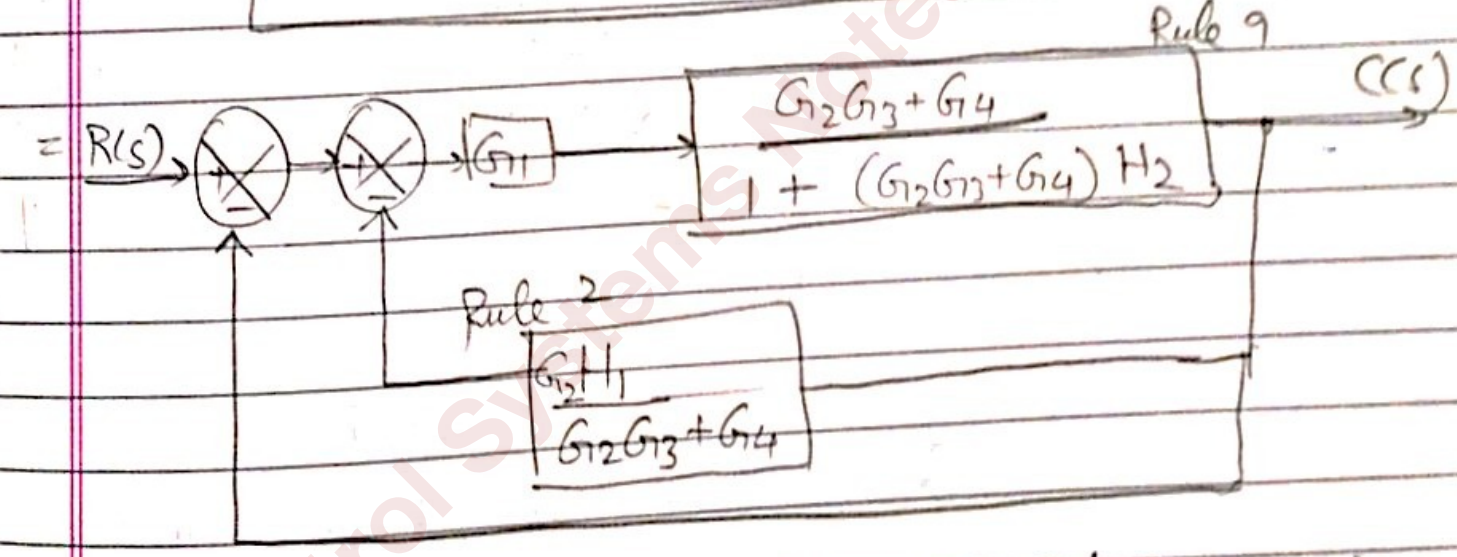
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Rule 2, 3

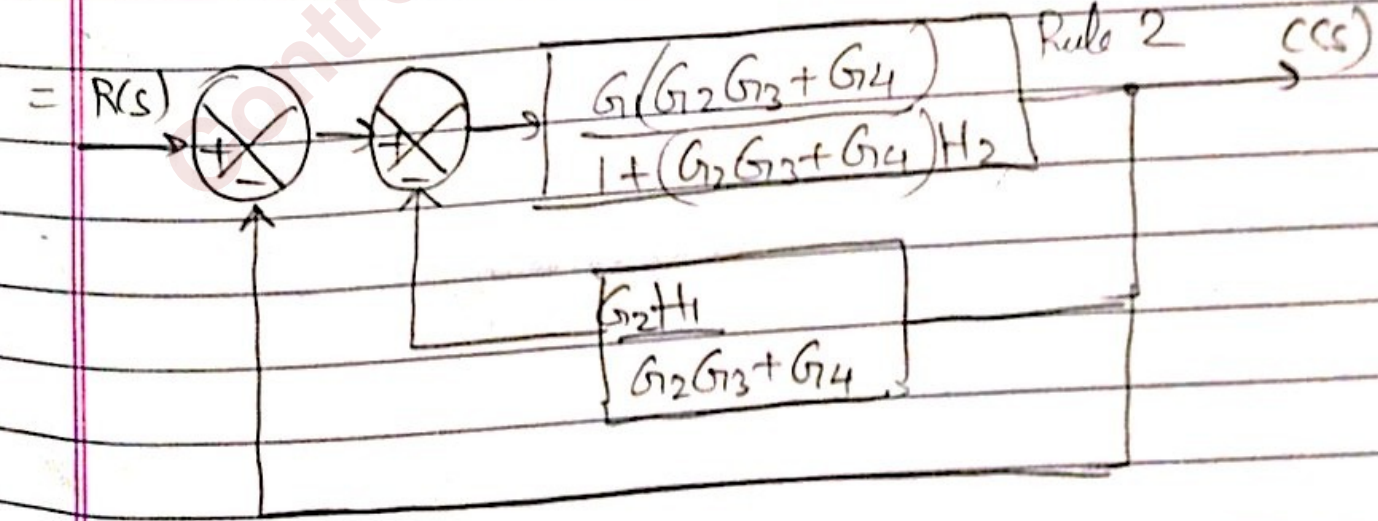




Rule 9

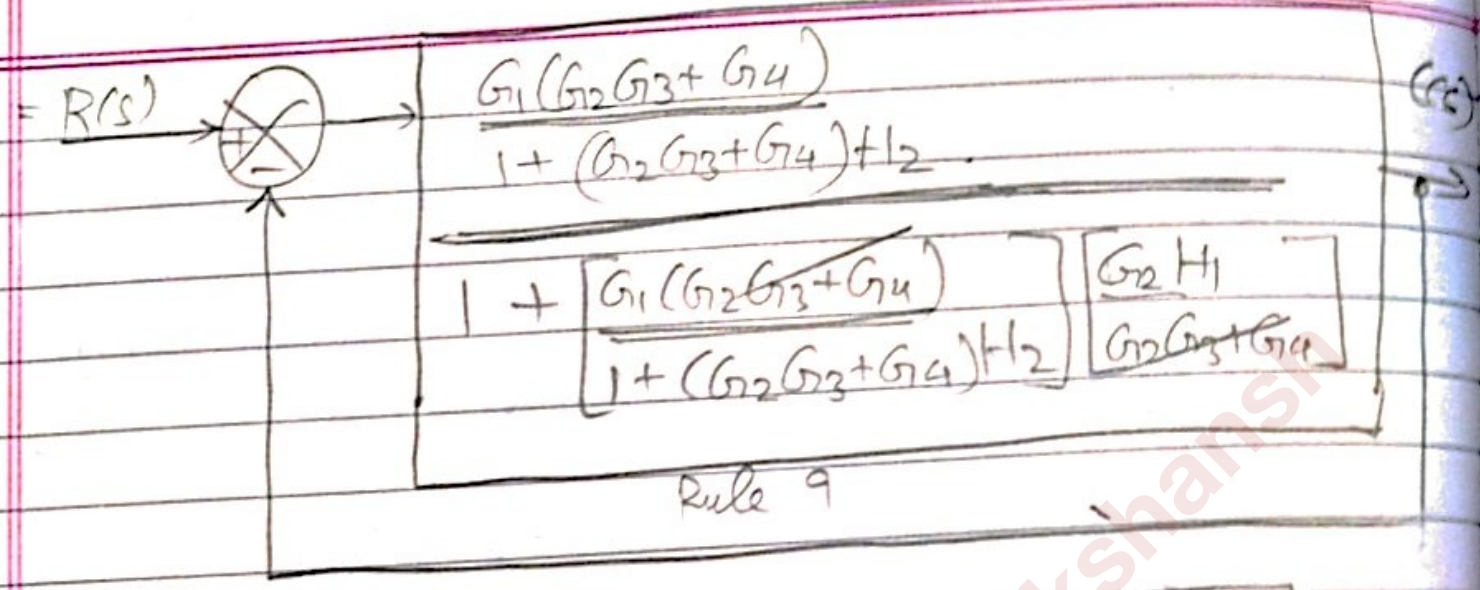


Rule 2

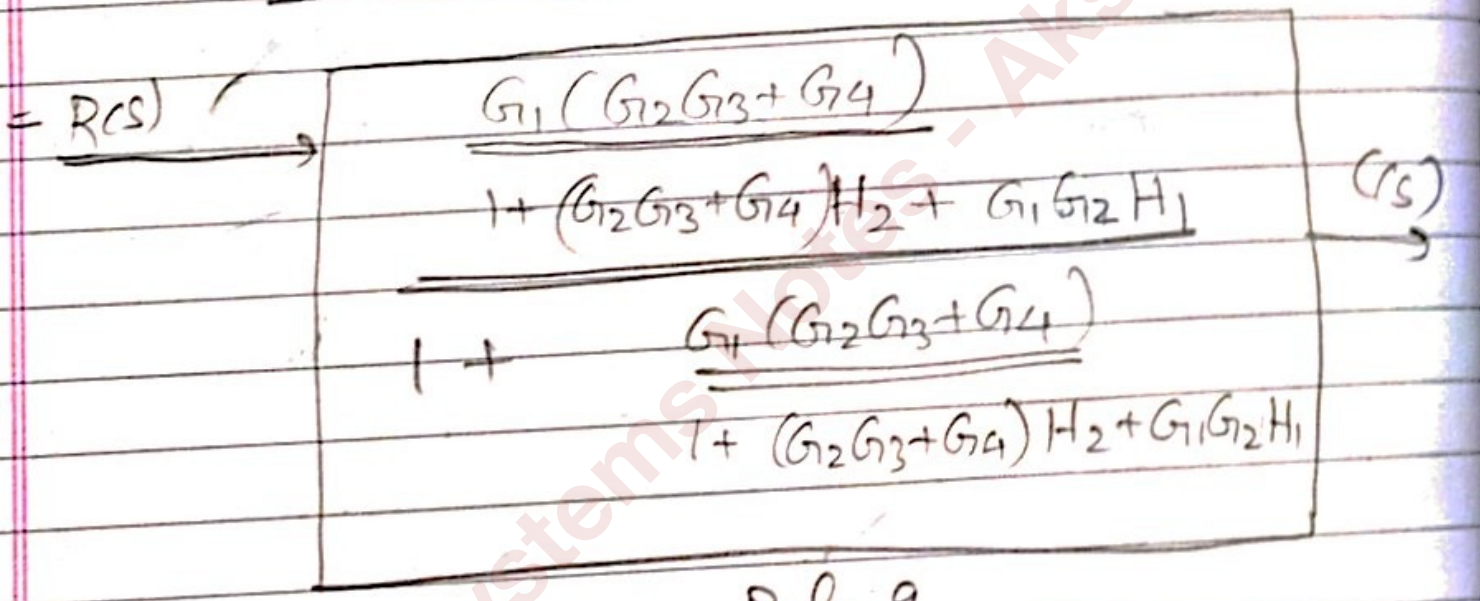


Control Systems Notes - Akshansh

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Rule 9



Rule 9

$$\Rightarrow TF = \frac{C(s)}{R(s)} = \frac{G_1(G_2G_3+G_4)}{1+(G_2G_3+G_4)H_2+G_1(G_2G_3+G_4)+G_1G_2H_1}$$

$$\Rightarrow TF = \frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1+(G_1+H_2)(G_2G_3+G_4)}$$

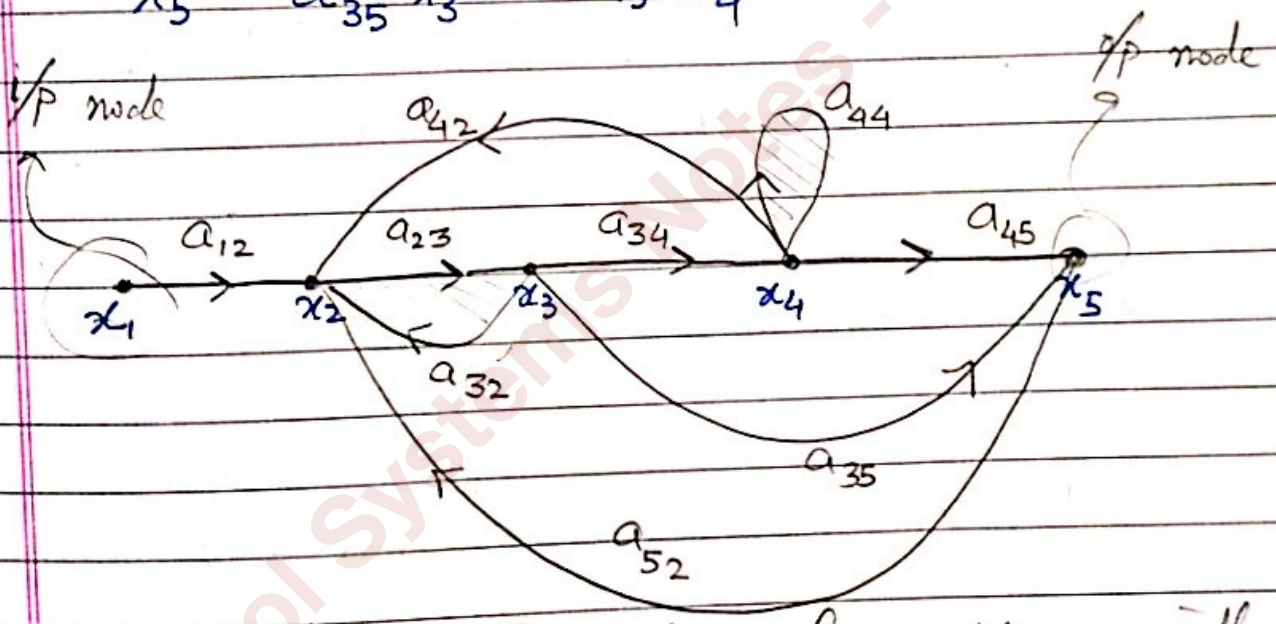
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SIGNAL FLOW GRAPH -

- Developed by S. J. Mason.
- Its a graphical represent^{n} b/w the variables of a set of linear algebraic eq^{ns}.

$$x_2 = a_{12}x_1 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5$$
$$x_3 = a_{23}x_2$$
$$x_4 = a_{34}x_3 + a_{44}x_4$$
$$x_5 = a_{35}x_3 + a_{45}x_4$$



- The given fig. shows how above eq^{ns} are written. The curves & lines indicate how many terms will exist for the variable (x_1, x_2 ...).
  eg. for x_3 → ∃ only 1 line.
  x_2 going to x_3 via a_{23}
  ∴ $x_3 = a_{23}x_2$ ✓

Control Systems Notes - Akshansh

* Mason's Gain formula :-

• Referring to previous pg. fig :-

S1. ∃ 2 forward paths with path gains

$P_1 = a_{12} a_{23} a_{34} a_{45}$ $\quad$ $P_2 = a_{12} a_{23} a_{35}$

i.e, from i/p node to o/p node, there exist only 2 ways/paths to reach.

∵ say, $P_1 = a_{12} a_{23} a_{34} a_{45}$

⇒ Go from a_{12} to a_{23} to a_{34} to a_{45} -

So, we reach from x_1 to x_5.

Note : The paths are traversed only once while writing Path gains.

S2. 5 individual loops with loop gains :-

$P_{11} = a_{23} a_{32}$ $\quad$ $P_{21} = a_{23} a_{34} a_{42}$

$P_{31} = a_{44}$ $\quad$ $P_{41} = a_{23} a_{34} a_{45} a_{52}$ → Path reaching back to x_2.

$P_{51} = a_{23} a_{35} a_{52}$

→ Imply that they reach the same pt. again (∃ individual loops)

S3. Two combin^{ns} of 2-non-touching loops.

$P_{12} = (a_{23} a_{32})\, a_{44}$ $\quad$ $P_{22} = (a_{23} a_{35} a_{52})\, a_{44}$

implies: 2 non touching loops → both the loops don't have any common node (shaded in prev. fig.)

Control Systems Notes - Akshansh

S4. Zero combin^{ns} of 3 non touching loops, 4 non touching loops etc.

—x—

~~FORMULA:~~

* Δ = 1 − {Σ of all individual loop gains}
  + {Σ of gain products of all possible combin^{ns} of 2 non touching loops}
  − {Σ of gain products of all possible combin^{ns} of 3 non touching loops}
  + {Σ of gain products of all possible combin^{ns} of 4 non touching loops}
  − − − − − −

(Δ → the denominator in the TF)

From the given fig,

$$\Delta = 1 - (a_{23}a_{32} + a_{23}a_{34}a_{42} + a_{44} + a_{23}a_{34}a_{52} + a_{23}a_{35}a_{52} + (a_{23}a_{32}a_{44} + a_{23}a_{35}a_{52}a_{44}) \rightarrow (1)$$

S5. First forward path (P_1) is in touch with all loops. So, $\Delta_1 = 1$ → ∵ ∃ no loop that is left untouched by the path on its way.

(Δ_1 → refers to the first path)

2nd forward path is not in touch with one loop

2nd path → $\Delta_2 = 1 - (a_{44})$ → a_{44} is left untouched by Path 2 (P_2)

Control Systems Notes - Rishabh

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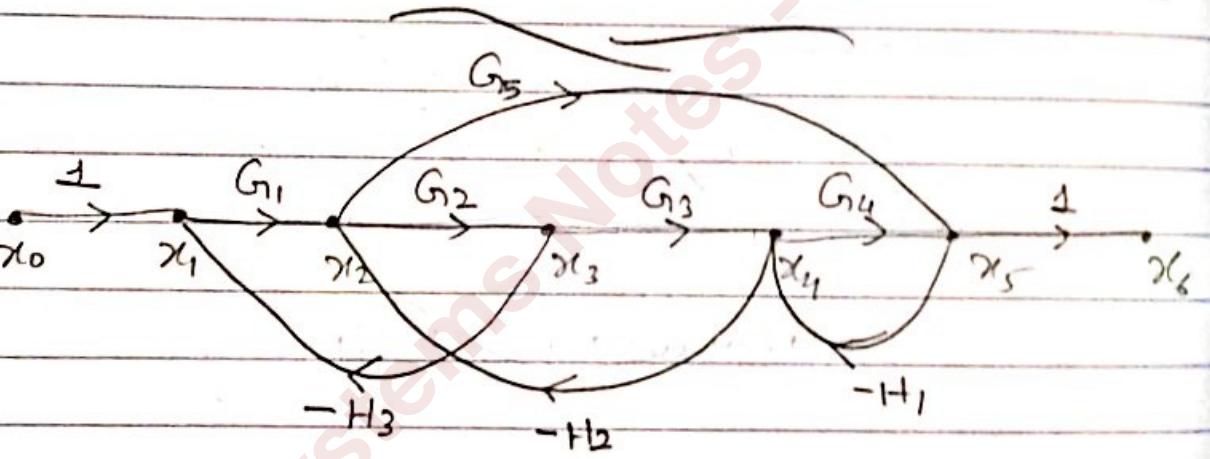
- FORMULA for TF.

$$\text{TF} = \frac{\text{o/p node}}{\text{i/p node}} = \frac{x_5}{x_1} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + \dots}{\Delta}$$

Here, $\text{TF} = \dfrac{(a_{12}a_{23}a_{34}a_{45})(1) + (a_{12}a_{23}a_{35})(1-a_{44})}{\Delta}$

Δ → from eq" (1)

(1)



S1) $P_1 = 1(G_1G_2G_3G_4)1 = G_1G_2G_3G_4$

$P_2 = 1(G_1G_5)\cdot 1 = G_1G_5$

S2) $P_{11} = G_1G_2(-H_3)$

$P_{21} = G_2G_3(-H_2)$

$P_{31} = G_5(-H_1)(-H_2)$

$P_{41} = -G_4H_1$

Control Systems Notes - Akshansh

S3) $P_{12} = (-G_1G_2H_3)(-G_4H_1)$

S4) No combin^n of 3 non touching loops, 4 non touching etc

S5) $\Delta_1 = 1$
$\Delta_2 = 1$
; $\Delta = 1 - [-G_1G_2H_3 - H_2G_2G_3 + H_1H_2G_5 - H_1G_4] + [G_1G_2G_4H_1H_3]$

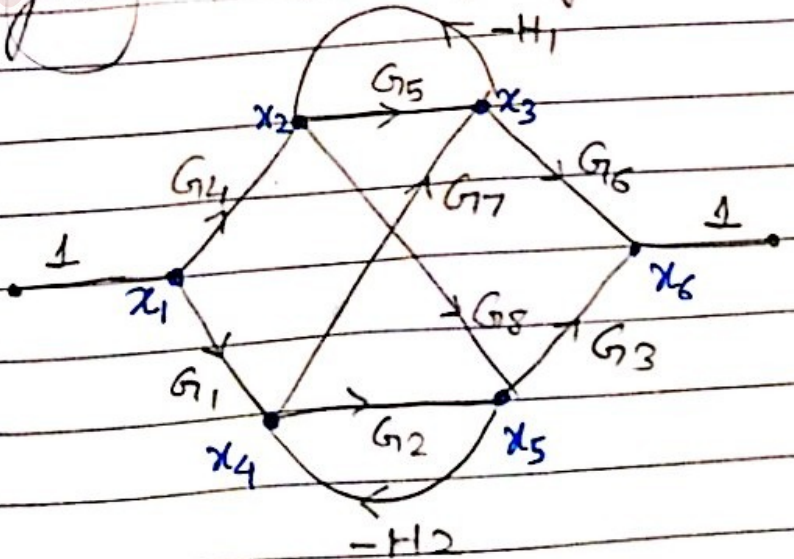
$$TF = \frac{P_1\Delta_1 + \Delta_2\Delta_2}{\Delta}$$

$$\Rightarrow TF = \frac{G_1G_2G_3G_4 + G_1G_5}{1 - (-G_1G_2H_3 - G_2G_3H_2 + G_5H_1H_2 - G_4H_1) + (G_1G_2G_4H_1H_3)}$$

$$TF = \frac{G_1G_2G_3G_4 + G_1G_5}{1 + G_1G_2H_3 + G_2G_3H_2 - G_5H_1H_2 + G_4H_1 + G_1G_2G_4H_1H_3}$$

TF

Q) Find gain of following sys. using Mason's formula



Control Systems Notes - Akshansh

S1) $P_1 = G_4 G_5 G_6$
$P_2 = G_4 G_8 G_3$
$P_3 = G_1 G_2 G_3$
$P_4 = G_1 G_7 G_8$
$P_5 = -G_4 G_8 G_7 G_6 H_2$
$P_6 = -G_1 G_7 G_8 G_3 H_1$

S2) Individual loops

$P_{11} = -G_2 H_2$
$P_{21} = -G_5 H_1$
$P_{31} = G_7 G_8 H_1 H_2$

S3) 2 non touching
$P_{12} = (G_5 H_1)(-G_2 H_2) = G_2 G_5 H_1 H_2$

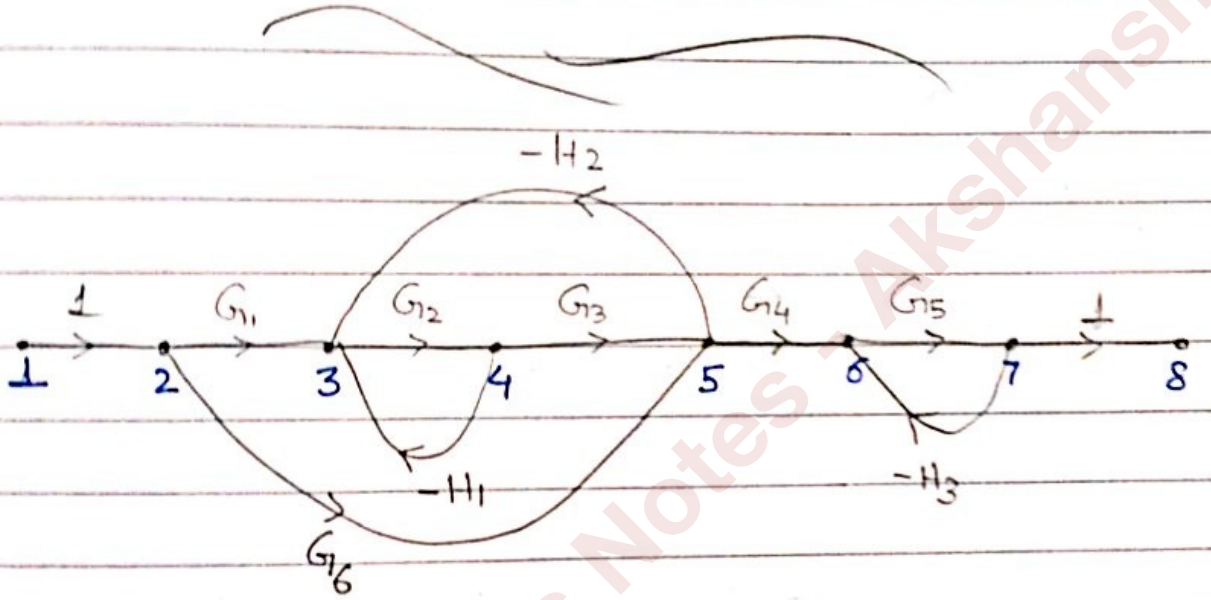
S4) $\Delta = 1 - [-G_2 H_2 - G_5 H_1 + G_7 G_8 H_1 H_2]$
$+ [G_2 G_5 H_1 H_2]$
$\Delta_1 = 1 + G_2 H_2$
$\Delta_3 = 1 + G_5 H_1$
$\Delta_2 = 1$
$\Delta_4 = 1$
$\Delta_5 = 1$
$\Delta_6 = 1$

$$TF = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6}{\Delta}$$

Control Systems Notes - Akshansh

$$TF = \frac{[G_4G_5G_6(1+G_2H_2)] + G_4G_8G_3 + [G_1G_2G_3(1+G_5H_1)] + G_{11}G_7G_6 - G_4G_6G_7G_8H_2 - G_1G_3G_7G_5H_1}{1 + G_2H_2 + G_5H_1 - G_7G_8H_1H_2 + G_2G_5H_1H_2}$$

Q.



S1) Path gains

$P_1 = G_1G_2G_3G_4G_5$: 12345678

$P_2 = G_6G_4G_5$ = 125678

S2) Individual loops

343 : $-G_2H_1$

3453 : $-G_2G_3H_2$

676 = $-G_5H_3$

S3) 2 non touching

(343)(676) = $G_2G_5H_1H_3$

(3453)(676) = $G_2G_3G_5H_2H_3$

Control Systems Notes - Akshansh

S4) $\Delta = 1 - [-G_2H_1 - G_2G_3H_2 - G_5H_3]$

$+ [G_2G_5H_1H_3 + G_2G_3G_5H_2H_3]$

$\Delta_1 = 1$

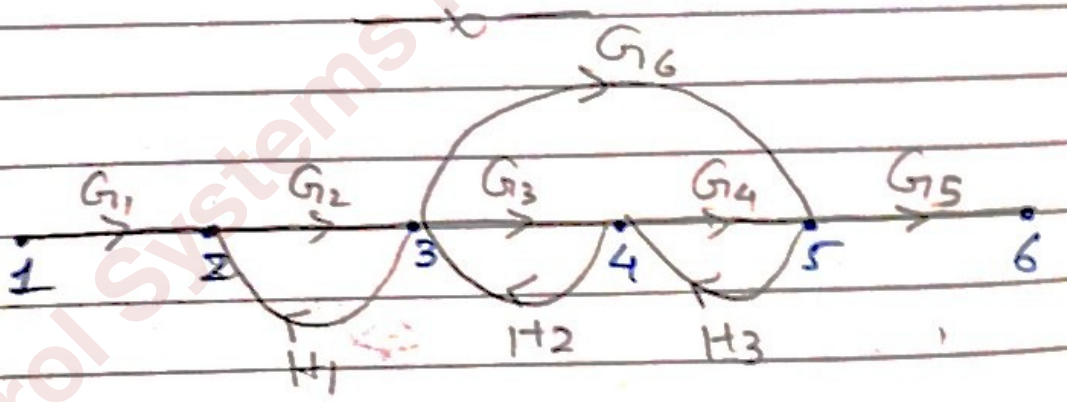
$\Delta_2 = 1 + G_2H_1$ (343 not included)

So, $TF = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$

$\Rightarrow TF = \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6(1 + G_2H_1)}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$

Ans

Q.



Path gains

S1) $12345\,6 = G_1G_2G_3G_4G_5$

$12356 = G_1G_2G_5G_6$

S2) Individual loops

$232 = G_2H_1$ $\quad 3543 = G_6H_2H_3$

$343 = G_3H_2$

$454 = G_4H_3$

Control Systems Notes - Akshansh

S3) 2 non touching

$(232)(454) = G_2 G_4 H_1 H_3$

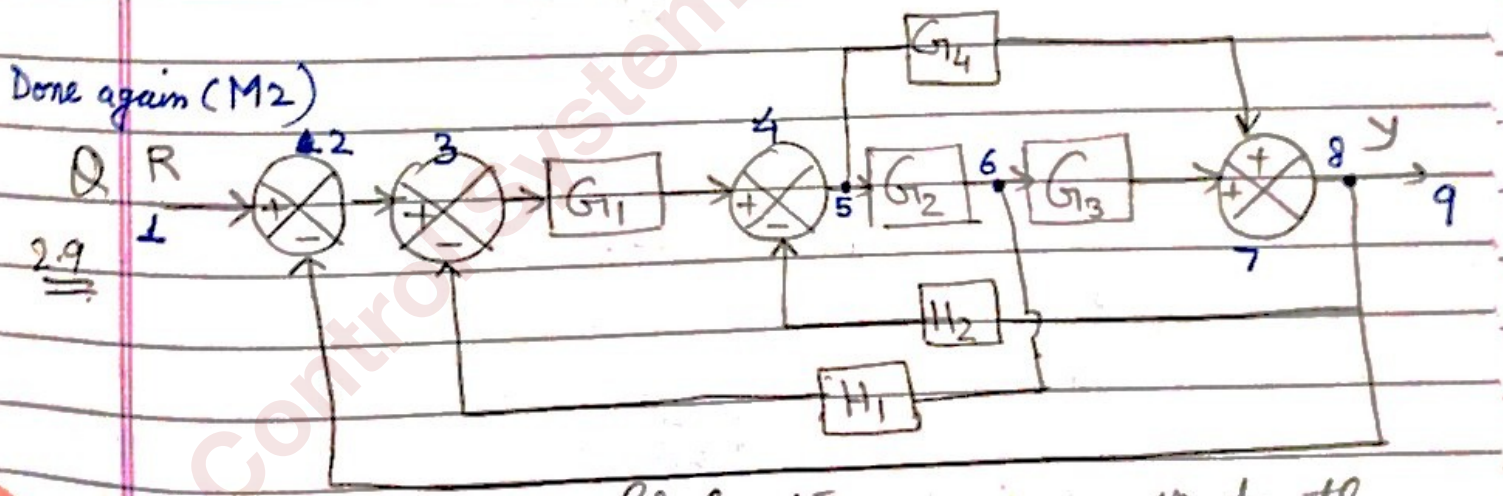
S4) $\Delta = 1 - [G_2H_1 + G_3H_2 + G_4H_3 + G_6H_2H_3] + [G_2G_4H_1H_3]$

$\Delta_1 = \Delta$

$\Delta_2 = 1$

$$TF = \frac{G_1G_2G_3G_4G_5 + G_1G_2G_5G_6}{1 - G_2H_1 - G_3H_2 - G_4H_3 - G_6H_2H_3 + G_2G_4H_1H_3}$$

Done again (M2)



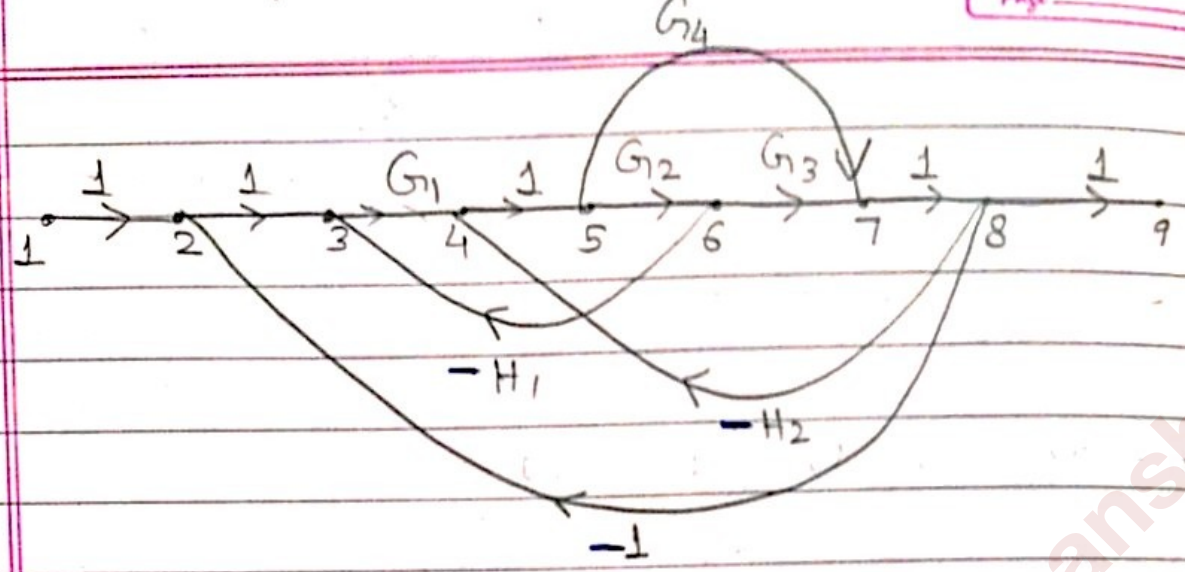
2.9

* Convert the given block diagram represent^n to the corresponding SFG & hence determine the TF.

S1) Number the i/p & o/p nodes. Number the summing pts. & take off pts. (shown above)

S2) Create SFG by connecting nodes & seeing their corresponding gains.

Control Systems Notes - Akshansh



S1) $P_1 = 12345678 = G_1G_2G_3$

$P_2 = 12345789 = G_1G_4$

S2) Individual loops

$P_{11} = (34563) = -G_1G_2H_1$ $\quad P_{41} = (456784) = -G_2G_3H_2$

$P_{21} = (45784) = -G_4H_2$ $\quad P_{51} = (2345782) = -G_1G_2G_3$

$P_{31} = (2345782) = -G_1G_4$

S3) 2 non touching

→ none ←

S4) $\Delta = 1 - (-G_1G_2H_1 - G_4H_2 - G_1G_4 - G_2G_3H_2 - G_1G_2G_3)$

$\Delta_1 = 1$

$\Delta_2 = 1$

$$TF = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_4H_2 + G_1G_4 + G_2G_3H_2 + G_1G_2G_3}$$

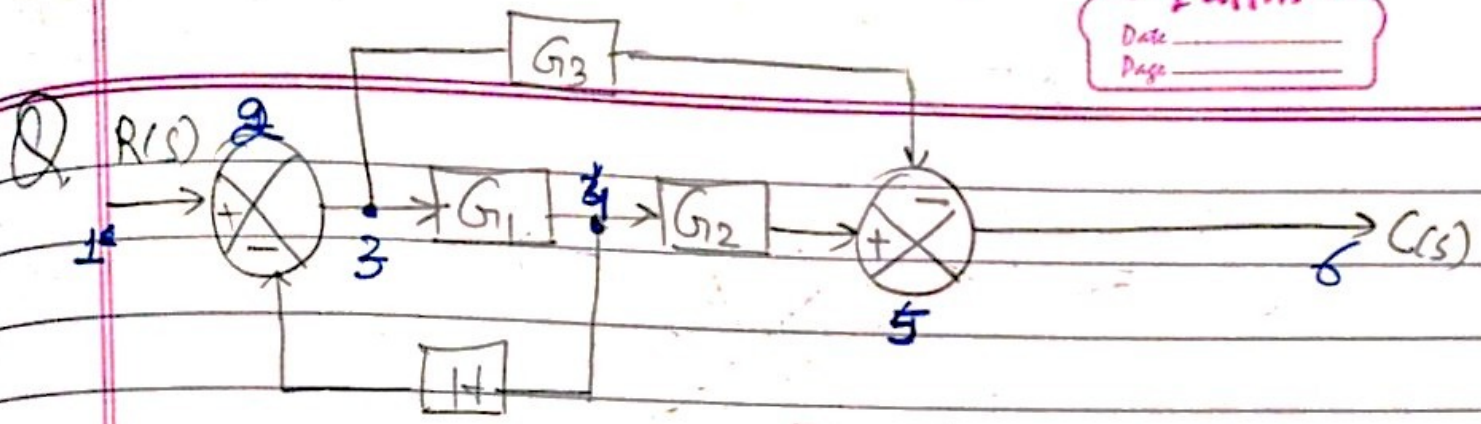
$$\Rightarrow TF = \frac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1 + (G_1 + H_2)(G_2G_3 + G_4)}$$

(same as before)

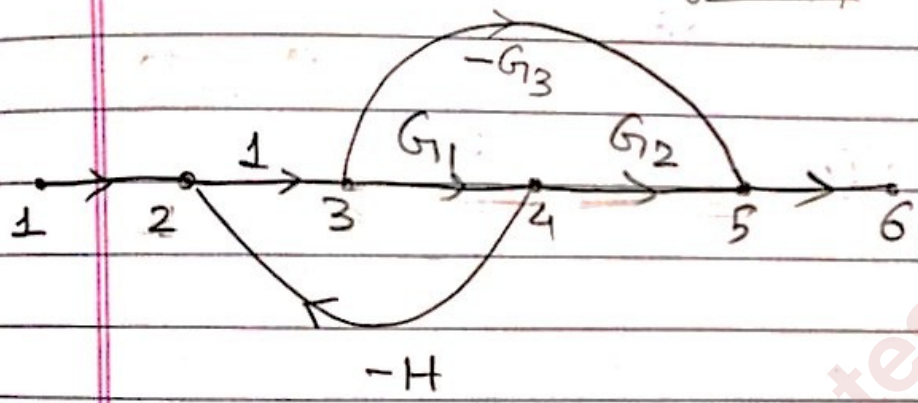
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Q.



Convert to SFG & find TF



S1) $P_1 = 123456 = G_1 G_2$

$P_2 = 123156 = -G_3$

S2) Individual loops

$= (2342) = -G_1 H$

S3) 2 non touching

(none)

S4) $\Delta = 1 - (-G_1 H)$

$\Delta_1 = 1$

$\Delta_2 = 1$

S5) $TF = \dfrac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$

$TF = \dfrac{G_1 G_2 - G_3}{1 + G_1 H}$

Control Systems Notes - Akshansh

* Robust : unaffected by external disturbance
  e.g: Robust health ≡ unaffected by virus, bacteria

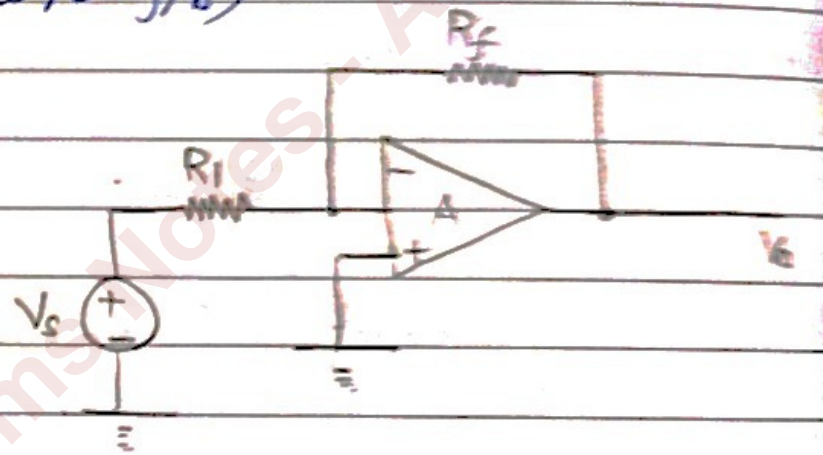
Chapter - 3

SENSITIVITY TO PARAMETER VARIATION UNDER FEEDBACK

eg $v_o = -A v_s$ (w/o f/b)

$v_o = -\frac{R_f}{R_1} v_s$

OP-AMP Circuit



Theoretically, its gain is ∞

Now, as R_f is giving -ve f/b to amplifier, so by the formula, $v_o = -\frac{R_f}{R_1} v_s$, the gain can be controlled instead of R_1 infinite. So, there goes stability.

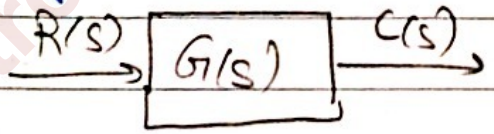
So, compromise gain to get stability ... & vice versa

- Feedback is used to reduce the diff. b/w reference i/p & controlled i/p.
  - The actuating signal (error signal) is applied to the process G(s) so as to influence output c in a manner which tends to reduce the error.
    - G(s) → also called process
    - c → controlled variable

Control Systems Notes - Akshansh

- The feedback not only reduces the error but also reduces the sensitivity of the sys. to parameter variations and unwanted internal & external disturbances.
- Parameter variations affect the system adversely.
- One of the primary purposes of using feedback is to reduce the sensitivity of sys. to parameter variations. The parameter may vary with age, changing envt. etc.
- Sensitivity: Its a measure of the effectiveness of feedback in reducing the influence of parameter variations on sys. performance.

✱ Effect of Parameter Variation in an open loop Control Sys:-



$$TF = \frac{C(s)}{R(s)} = G(s)$$

$$\Rightarrow C(s) = G(s) \cdot R(s)$$

If $\Delta G(s)$ is the change in $G(s)$ due to parameter variation,

$$C(s) + \Delta C(s) = [G(s) + \Delta G(s)]\, R(s)$$

$$\Rightarrow C(s) + \Delta C(s) = G(s)R(s) + \Delta G(s) \cdot R(s)$$

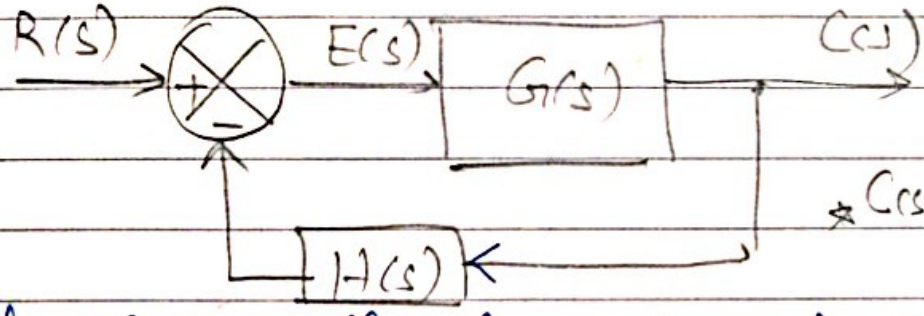
$$\Rightarrow \Delta C(s) + C(s) = C(s) + \Delta G(s) \cdot R(s)$$

$$\Rightarrow \Delta C(s) = \Delta G(s) \cdot R(s) \longrightarrow (1)$$

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☆ Effect of parameter variations in closed loop control:



$$\star C(s) = \left[\frac{G(s)}{1+H(s)}\right] R(s)$$

If $\Delta G(s)$ is the change in $G(s)$ due to parameter variation, corresponding change in o/p

$$C(s) + \Delta C(s) = \left[\frac{G(s) + \Delta G(s)}{1 + G(s)H(s) + \Delta G(s)\Delta H(s)}\right] R(s)$$

$$\because \quad C(s) = \left[\frac{G(s)}{1+G(s)H(s)}\right] R(s)$$

Now $|G(s)| >> |\Delta G(s)|$

$$\text{i.e.,} \quad C(s) + \Delta C(s) = \left[\frac{G(s) + \Delta G(s)}{1 + G(s)H(s)}\right] R(s)$$

$$\text{i.e.,} \quad C(s) + \Delta C(s) = \frac{R(s)G(s)}{1+G(s)H(s)} + \frac{\Delta G(s)\,R(s)}{1+G(s)H(s)}$$

$$\Rightarrow C(s) + \Delta C(s) = C(s) + \frac{\Delta G(s)\,R(s)}{1+G(s)H(s)}$$

$$\Rightarrow \Delta C(s) = \left[\frac{\Delta G(s)}{1+G(s)H(s)}\right] R(s) \longrightarrow (2)$$

Comparing (1) & (2), the change in o/p of closed loop sys due to variation $G(s)$ is reduced by a factor of $1 + G(s)H(s)$

$|1 + G(s)H(s)| >> 1$ in most cases over frequency ($s = j\omega$) of interest

Control Systems Notes - Akshansh

Sensitivity of a Control System

- Sensitivity : relative variation in the overall TF $T(s) = C(s)/R(s)$ due to variation in $G(s)$

$$\text{Sensitivity} = \frac{\% \text{ change in } T(s)}{\% \text{ change in } G(s)}$$

↳ For small incremental variation in $G(s)$, the sensitivity is written in its quantitative form as,

$$S_G^T = \frac{\partial T/T}{\partial G/G} = \frac{\partial \ln T}{\partial \ln G} \longrightarrow (3)$$

↳ S_G^T denotes the sensitivity of T w.r.t G.

↳ $S_G^T = \frac{\partial T}{\partial G} \times \frac{G}{T}$ (from (3))

Now, $T = \frac{G}{1+GH}$

$$\Rightarrow \frac{\partial T}{\partial G} = \frac{(1+GH) - G(H)}{(1+GH)^2}$$

$$\Rightarrow \frac{\partial T}{\partial G} = \frac{1}{(1+GH)^2}$$

Now $\frac{\partial T}{\partial G} \times \frac{G}{T} = \frac{\partial T}{\partial G} \times \frac{G \times (1+GH)}{G}$

$$\Rightarrow S_G^T = \frac{1}{1+GH} \longrightarrow (8)$$

↳ Sensitivity for open loop sys,

$$S_G^T = \frac{\partial T}{\partial G} \times \frac{G}{T} = 1 \quad (\because T = G)$$

$\left(\frac{\partial T}{\partial G} = \frac{\partial}{\partial G}(G) = 1 \text{ & } \frac{G}{T} = 1\right)$

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∴ The sensitivity of a closed loop sys w.r.t variation in G is reduced by a factor of (1+GH) as compared to that of open loop system.

- **Sensitivity of T w.r.t H** $\left(T = \frac{G}{1+GH}\right)$

$$S_H^T = \frac{\partial T}{\partial H} \times \frac{H}{T} = \left[\frac{(1+GH)0 - G(G)}{(1+GH)^2}\right] \times \frac{H}{T}$$

$$= -\frac{G^2}{(1+GH)^2} \times \frac{H(1+GH)}{G}$$

$$\Rightarrow S_H^T = \frac{-GH}{1+GH} \longrightarrow (A).$$

& from prev. pg, $S_G^T = \frac{1}{1+GH} \longrightarrow (B)$.

From (A) & (B), it is concluded that closed loop sys. is more sensitive to variations in feedback sensor H than variations in G.

$$S_H^T = \frac{-GH}{1+GH}$$

↳ for $GH >>> 1$, $S_H^T \to 1$.

Changes in H directly affect the sys. o/p. So, it is imp. to use feedback elements that do not vary with environmental changes.

☆ So, feedback leads to less changes in sensitivity. So, its advantageous.

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Self

☆ Read section reading some applic^n of G(s) & H(s) from slides.

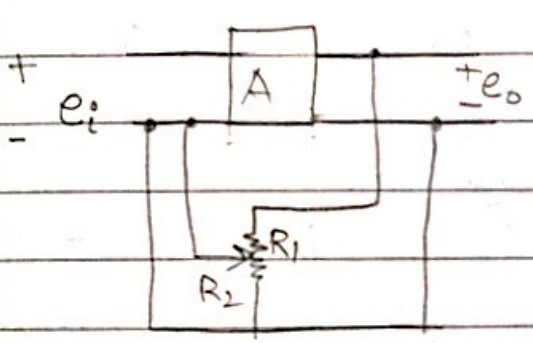
☆ For an open loop sys, gain = G(s)
& for a closed loop sys, that gain = $\frac{G(s)}{1+H(s)G(s)}$
So, in a closed loop sys, gain is reduced by a factor of $\frac{1}{1+GH}$. Or, we can say that the gain was sacrificed in order to get more stability (∵ we know that closed loop sys is less sensitive & hence more stable as compared to open loop sys)

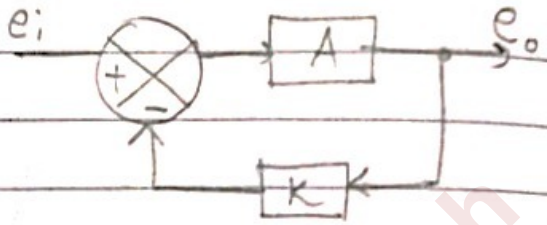
→☆ Closed loop sys, G(s) may be less rigidly specified because the effects of parameter variations are checked by feedback.
But for H(s) – components are carefully selected.

→☆ G(s) is made of power elements, ~~H(s)~~
☆ H(s) is made of measuring elements (sensor) that operate at low power levels.
Selection of H(s) is far less costly than selection of G(s).

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ex Show that feedback ↓ System's sensitivity to parameter variations





Feedback Amplifier

Potential divider provides a -ve feedback of gain ($K < 1$)

Assume:

↳ I/p impedance of amplifier $\to \infty$

O/p " $\to 0$

$$T = \frac{e_o}{e_i} = \frac{A}{1+kA}; \quad K = \frac{R_2}{R_1} \leq 1.$$

$$S_A^T = \frac{dT}{dA} \times \frac{A}{T} = \frac{1}{1+KA}.$$

For $A = 10^4$; $K = 0.1$; $S_A^T = 0.001$

Overall gain $T = \frac{10^4}{1+(0.1)(10^4)} = 9.99 \simeq 10.$

So, it's seen that

- f/b reduces sensitivity to variation in forward gain (A) to 0.001
- f/b also reduces overall gain from 10^4 to 9.99.

• Sensitivity to f/b gain K,

$$S_K^T = \frac{dT}{dk} \cdot \frac{k}{T} = \frac{-kA}{1+kA}.$$

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$$= \frac{-(0.1)(10^4)}{1+(0.1)(10^4)} \approx -1.$$

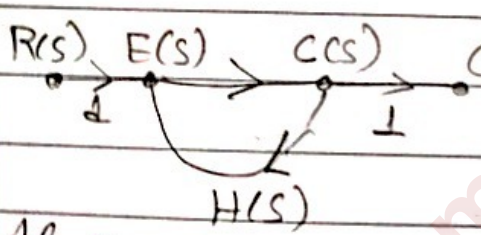
This indicates $k = \frac{R_2}{R_1}$ must be constant

For a large A, $kA >>> 1$.

$$T = \frac{A}{kA} \approx \frac{1}{k} \approx 10 \approx \frac{R_1}{R_2} = \text{independent of A}$$

✱ REGENERATIVE f/b. (+ve f/b)

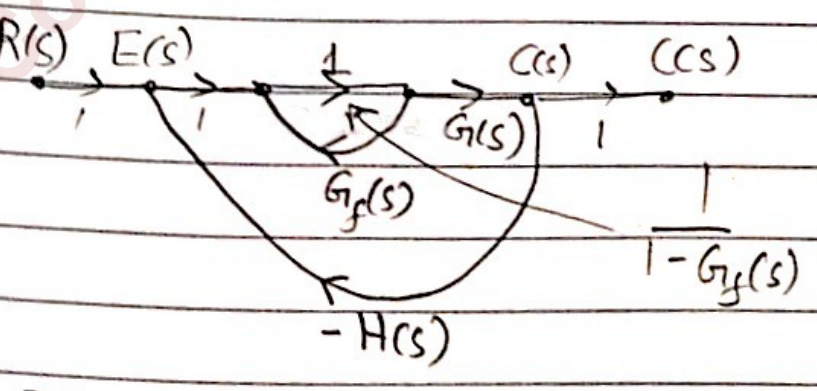
The o/p is ~~+ve~~ with +ve sign. (f/b)



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

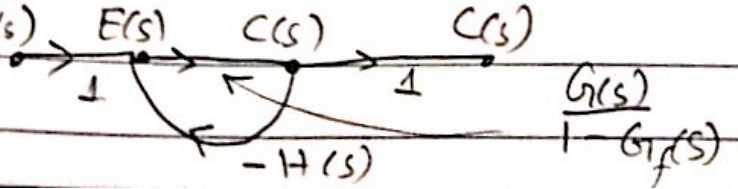
If $G(s)H(s) \to 1$, $\frac{C(s)}{R(s)} \to \infty$ → Infinite o/p for a finite i/p.
→ Instability cond^n

Regenerative f/b: sometimes used for ↑ loop gain of sys.



Loop gain $= \frac{-G(s)H(s)}{1 - G_f(s)}$

If $G_f(s) \simeq 1$; loop gain $\to \infty$



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for closed loop TF :-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G_f(s) + G(s)H(s)} \simeq \frac{1}{H(s)}$$

$\simeq$ is insensitive to $G(s)$ due to high loop gain provided by inner regenerative f/b loop.

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Chapter - 5

TIME RESPONSE ANALYSIS

* **Time Response:**
  Response given by the sys. which is a fn of time to the applied excitation is called time response.

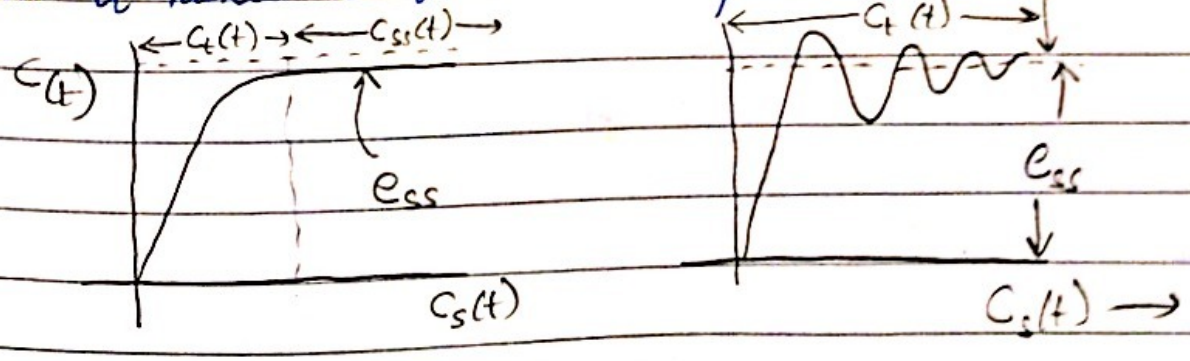
  Shows 2 behaviour or, is tested for

  Transient behaviour → Steady State behaviour

* For any specific i/p signal, a complete time response expression can be obtained by inverse L.T. This expression yields steady state (SS) behaviour of the sys. with $t \to \infty$.
* Transient response analysis: need to be done only if sys. is stable.

  : obtained by applying ss i/p.

* **TRANSIENT RESPONSE:** The o/p variation during the time it takes to achieve its final value.



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SS Response:

* Response of a sys as $t \to \infty$ from which the time at which transient response dies completely

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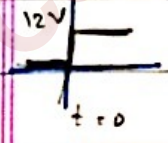
- $C(t) = C_t(t) + C_{ss}(t)$
- C_{ss}: steady state ~~or~~ error
- $C_t(t)$: exponential.

* Info. from Transient Response:
  - When the sys has started showing its response to applied excitation.
  - Rate of rise of o/p
  - o/p ↑ exponentially or oscillating
  - o/p settling time at final value.
  - If o/p is oscillating, overshooting at final value.

* Time Response Characteristics.
  - → Transient: only if sys. is stable.
  - → Steady state (ss): when the sys. is at SS.

* Std. Test signals → for testing cs
  - Step
  - Ramp
  - Parabolic
  - Impulse

(unit step fn) $u(t)$



* Info. from SS response:
  - S.S error
  - error is const or varying with time.

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* STEP INPUT:

A signal whose value changes from one level (usually zero) to another level 'A' in zero time.

Defined as:-

$$r(t) = A\,u(t)$$

→ $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

→ $u(t)$: unit step fn.

→ $\mathcal{L}(r(t)) = \mathcal{L}(A\,u(t))$

$= A\,\mathcal{L}(u(t))$ ... 1

$= \frac{A}{s}$



* RAMP INPUT (vel. fn)

It is a signal which starts at a value of zero & ↑ linearly with time.

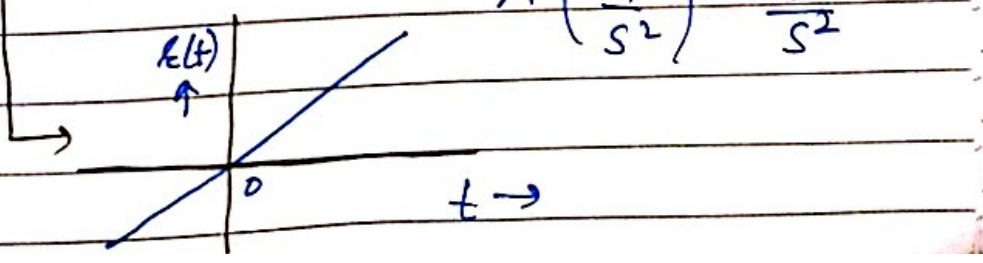
$$r(t) = \begin{cases} At & ; t > 0 \\ 0 & ; t < 0 \end{cases}$$

→ when $A = 1$; its UNIT RAMP.

→ $\mathcal{L}(r(t)) = \mathcal{L}(At)$

$= A\,\mathcal{L}(t)$

$= A\left(\frac{1}{s^2}\right) = \frac{A}{s^2}$



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★ $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

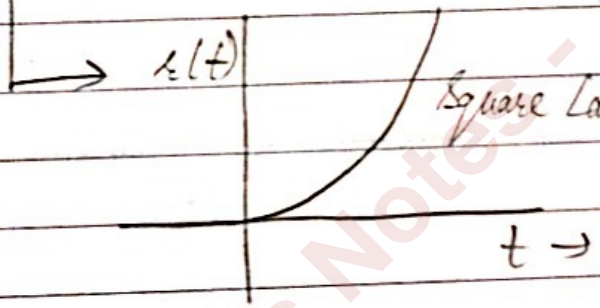
★ PARABOLIC INPUT (Accl^n fn)

$$r(t) = \begin{cases} \frac{A}{2}t^2 & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

$$\rightarrow \mathcal{L}(r(t)) = \mathcal{L}\left(\frac{A}{2}t^2\right)$$

$$= \frac{A}{2}\mathcal{L}(t^2)$$

$$\Rightarrow R(s) = \frac{A}{2}\frac{2!}{s^{2+1}} = \frac{A}{s^3}$$



★ IMPULSE SIGNAL (Direc Delta fn)

A unit impulse is defined as a signal which has zero value everywhere except at $t=0$, where its magnitude is infinite.

Also called δ fn

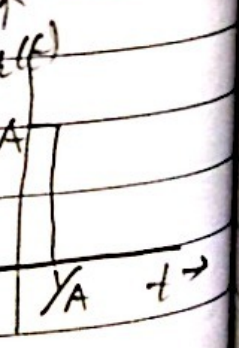
$$\delta(t) \begin{cases} = 0 & , t \neq 0 \\ \rightarrow \infty & , t = 0 \end{cases}$$

$$\rightarrow \int_{-\epsilon \text{ or } -\infty}^{\epsilon \text{ or } \infty} \delta(t)\,dt = 1$$

In practice, $\delta(t) = \frac{d}{dt}u(t)$ or $\dot{u}(t)$

& $\mathcal{L}(\delta(t)) = 1 = R(s)$

Reason (top)



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$$\mathcal{L}(\delta(t)) = \mathcal{L}\left(\frac{du(t)}{dt}\right)$$
$$= s\,\mathcal{L}(u(t)) - u(0)$$
$$= s\left(\frac{1}{s}\right) - 0$$
$$= 1.$$

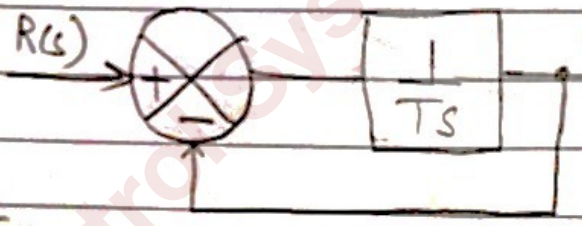
Impulse response of a sys with transfer f^n $\frac{C(s)}{R(s)} = G(s)$.

$$C(s) = G(s)R(s) = G(s)$$
$$C(s) = L^{-1}(G(s)) = g(t)$$
= Inverse LT of TR – weighting f^n.

– used to find system's response to any i/p $x(t)$ by convolution integral.

$$C(t) = \int_0^t g(t-\tau)\,x(\tau)\,d\tau$$

✱ Time Response of Ist Order Systems :



$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1} = T(s)$$

$$\Rightarrow C(s) = \frac{R(s)}{Ts+1}$$

For unit step i/p
$x(t) = 1$
$\Rightarrow R(s) = \frac{1}{s}$ (LT)

$$\Rightarrow C(s) = \frac{1}{s(Ts+1)}$$

or, do partial fraction

$$\Rightarrow C(s) = T\left(\frac{1}{Ts(Ts+1)}\right)$$
$$= T\left(\frac{(Ts+1) - Ts}{(Ts)(Ts+1)}\right)$$
$$\Rightarrow C(s) = \frac{T}{Ts} - \frac{T}{Ts+1}$$

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$\star \mathcal{L}[e^{-at}] = \frac{1}{s+a}$

$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

$\Rightarrow C(s) = \frac{1}{s} - \frac{T}{Ts+1}$

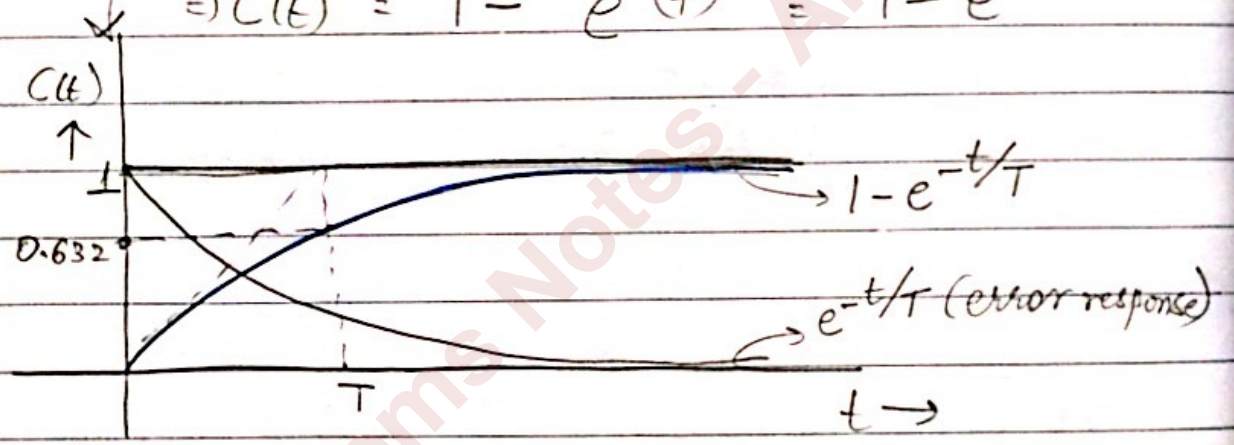
Now, $C(t) = \mathcal{L}^{-1}[C(s)]$

$= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{T}{Ts+1}\right]$

$= 1 - \mathcal{L}^{-1}\left[\frac{T}{T(s+\frac{1}{T})}\right]$

$= 1 - \mathcal{L}^{-1}\left[\frac{1}{s+1/T}\right]$

$\Rightarrow C(t) = 1 - e^{-(\frac{1}{T})t} = 1 - e^{-t/T}$



t	C(t)
0	0
T	0.632
2T	0.865
3T	0.95
:	:
∞	1

; T: time constt for a physical time → time when the fn has a value of 63% of its total value (1).

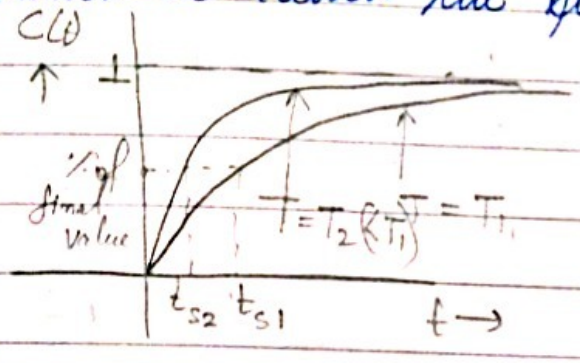
★ Unit Step response of Ist order Sys:-

- o/p rises exponentially to final value of 1.
- Initial slope of a curve at $t=0$

$$\left.\frac{dc}{dt}\right|_{t=0} = \frac{1}{T} e^{-t/T}\Big|_{t=0} = \frac{1}{T}$$; T: time constt.

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- Time constt. is indicative of how fast the sys tends to reach the final value.



Error response $e(t)$
$= r(t) - c(t) = e^{-t/T}$

Steady state error :-
$e_{ss} = \lim_{t \to \infty} e(t) = 0$

✱ Response to unit ramp i/p

Response of 1st ord. sys. to ramp i/p,

$$r(t) \begin{cases} = t \; ; t > 0 \\ = 0 \; ; t < 0 \end{cases} \; ; \; R(s) = \frac{1}{s^2}$$

$$C(s) = R(s) \cdot G(s) = \frac{1}{s^2} \left(\frac{1}{Ts+1} \right)$$

(Proved before)

$$\Rightarrow \frac{1}{s^2(Ts+1)} = \frac{As+B}{s^2} + \frac{C}{Ts+1}$$

$$\Rightarrow (As+B)(Ts+1) + Cs^2 = 1$$

Put $s = 0 \Rightarrow B = 1$

$s = -1/T \Rightarrow C = T^2$

$s = 1/T \Rightarrow \left(\frac{A}{T} + 1 \right)(2) + 1 = 1$

$\Rightarrow A = -T.$

$$\Rightarrow C(s) = -\frac{Ts+1}{s^2} + \frac{T^2}{Ts+1}$$

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$$\Rightarrow C(s) = \left[\frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1}\right]$$

$$\Rightarrow c(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1}\right]$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - T\left(\mathcal{L}^{-1}\left(\frac{1}{s}\right)\right) + \mathcal{L}^{-1}\left(\frac{T^2}{T(s+1/T)}\right)$$

$$\Rightarrow c(t) = t - T(1) + T e^{-t/T}$$

$$\left(\because \mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}\right)$$

$$\Rightarrow c(t) = t - T(1 - e^{-t/T}) \quad \rightarrow (1)$$

Now,

error signal,

$$e(t) = r(t) - c(t)$$

$$= t - [t - T(1 - e^{-t/T})]$$

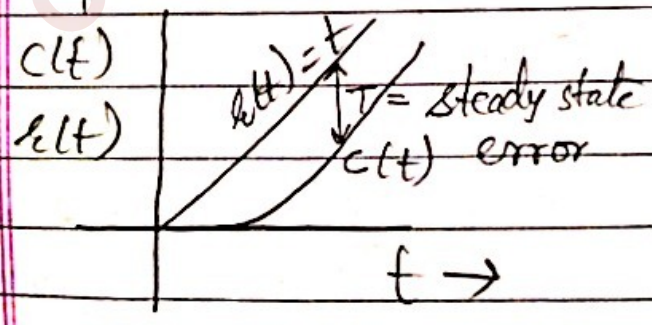
$$\Rightarrow e(t) = T(1 - e^{-t/T})$$

Now,

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} T(1 - e^{-\infty}) = T$$

$$\Rightarrow e_{ss} = T \text{ (Time Constt)}$$

M2 to find e_{ss}



$\because e_{ss} = T$,

↓ in T, increases speed of response & also ↓ e_{ss}

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✡✡ NOTE:

* Ramps signal : Integral of step signal.
* Parabolic signal : " " ramps signal
* Impulse " : Derivative of step fn

From (1),

$$c(t) = t - T(1 - e^{-t/T})$$

$$\Rightarrow \dot{c}(t) = 1 - e^{-t/T}$$

= Identical to sys. response to unit step i/p.

- Thus, transient response to unit ramps yields no addⁿᵃˡ info about speed of response.
- SS error to ramps i/p obtained directly by final value theorem:-

$$e_{ss} = \lim_{t \to 0} e(t) = \lim_{s \to 0} sE(s)$$

$$= \lim_{s \to 0} s[R(s) - C(s)]$$

$$= \lim_{s \to 0} s\left[\frac{1}{s^2} - \frac{1}{s^2(Ts+1)}\right]$$

$$= \lim_{s \to 0} \left[\frac{Ts+1-1}{s^2}\right]$$

$$\Rightarrow e_{ss} = \lim_{s \to 0} s\left[\frac{Ts}{s^2}\right] = T. \quad \text{(Same ans. as before)}$$

- This avoids the need to obtain inverse LT ($\mathcal{L}^{-1}$) (considerable labor saving in higher order sys)

Control Systems Notes - Akshansh

★ Zeta : ζ (Greek Symbol)
↳ Damping factor

Second Order Sys : Time Response

$$a_2 \frac{d^2c(t)}{dt^2} + a_1 \frac{dc(t)}{dt} + a_0 c(t) = b_0 r(t)$$

Taking LT

$$\Rightarrow a_2 C(s)[s^2] + a_1 C(s) \cdot s + a_0 C(s) = b_0 R(s)$$

$$\Rightarrow C(s)[a_2 s^2 + a_1 s + a_0] = b_0 R(s)$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{b_0}{a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0}{a_0\left(\frac{a_2}{a_0} s^2 + \frac{a_1}{a_0} s + 1\right)}$$

$$= \frac{b_0/a_0}{\frac{a_2}{a_0} s^2 + \frac{a_1}{a_0} s + 1}$$

$$= \frac{k}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1}$$

ζ → Zeta; Damping factor / ratio

ω_n → natural frequency of oscill^n.

Now, $k = 1$ (assumed) : GAIN Const^t

$$\Rightarrow \boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$ ★

↳ TF of std. 2nd ord. sys

↳ Denominator :

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \longrightarrow (1)$$

↳ Characteristic eq^n.

↳ 2 roots ; called as POLES

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Note: $\mathcal{L}^{-1}(aF(s)) = a\mathcal{L}^{-1}(F(s)) = af(t)$

; a : const

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* For the given TF for 2nd ord. sys

↳ $\zeta = 0$: undamped sys

$0 < \zeta < 1$: underdamped sys.

$\zeta = 1$: critically damped

$\zeta > 1$: overdamped.

—o—

* Finding roots of eqn (1),

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0.$$

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4(1)(\omega_n^2)}}{2(1)}$$

$$= \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}$$

$$\Rightarrow \boxed{s_1, s_2 = -\zeta\omega_n \pm \omega_n(\sqrt{\zeta^2 - 1})}$$

↳ $\zeta = 0$: $s_1, s_2 = \pm\omega_n\sqrt{-1} = \pm j\omega_n$.

↳ complex conjugate pairs.

↳ zero real parts.

↳ purely oscillatory.

↳ $\zeta = 1$; $s_1, s_2 = -\omega_n$

↳ roots are real, equal, -ve.

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for $\zeta = 2$, say

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$\sqrt{2^2+1}$

→ $\zeta > 1$: $s_1, s_2 = -2\omega_n \pm \omega_n \sqrt{3}$

→ roots are real, unequal, -ve.

→ $0 < \zeta < 1$: $s_1, s_2 = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$

→ roots are complex conjugates with -ve real parts.

→ for $\zeta = 0.5$, say,

$s_0, s_1, s_2 = -0.5\omega_n \pm \omega_n \sqrt{0.25 - 1}$

$= -0.5\omega_n \pm j\omega_n \sqrt{0.75}$

✱ Generally,

$s_1, s_2 = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$

$= -\zeta\omega_n \pm \omega_n \sqrt{(-1)(1-\zeta^2)}$

$\Rightarrow \boxed{s_1, s_2 = -\zeta\omega_n \pm j\omega_d}$ ✱

→ ω_d : damped frequency of oscill^n

$= \omega_n \sqrt{1-\zeta^2}$

CASE (I) $\zeta = 0$:

RESPONSE OF UNDAMPED SYS (2nd ord) for UNIT STEP i/p

→ $r(t) = 1$.

We know, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

for unit step i/p, $r(t) = 1 \Rightarrow R(s) = \frac{1}{s}$

$\Rightarrow C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ → 0

$\Rightarrow C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + \omega_n^2}$

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$\cos 2\theta = 1 - 2\cos^2\theta$

$\Rightarrow 2\cos^2\theta = 1 - \cos 2\theta$

$\Rightarrow$

$$\Rightarrow C(s) = \frac{\omega_n^2 + s^2 - s^2}{s(s^2+\omega_n^2)}$$

$$= \frac{\omega_n^2 + s^2}{s(s^2+\omega_n^2)} - \frac{s^2}{s(s^2+\omega_n^2)}$$

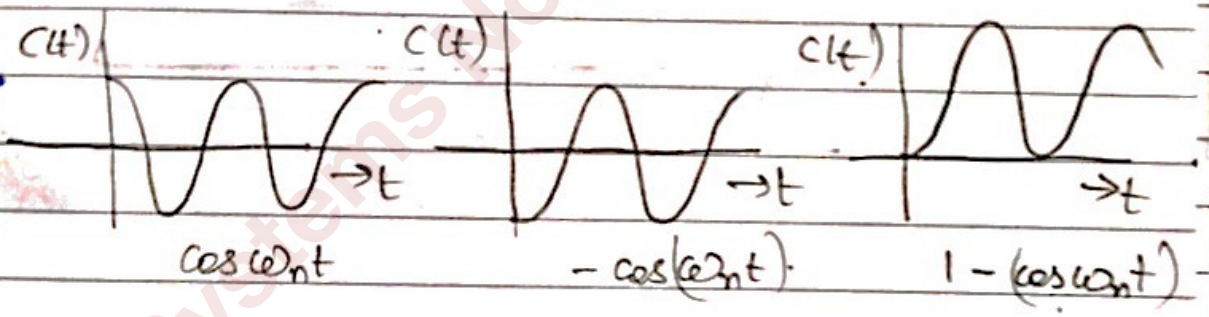
$$\Rightarrow C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$ (M2 : use Partial fraction)

Now, applying $\mathcal{L}^{-1}$, both sides.

$$\Rightarrow c(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+\omega_n^2}\right)$$

$$\Rightarrow \boxed{c(t) = 1 - \cos(\omega_n t)}$$

or $c(t) = 2\cos^2\left(\frac{\omega_n t}{2}\right)$: by formula



★ Note :- Some Useful Results on L^{-1}

1) $\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

2) $\mathcal{L}^{-1}\left(\frac{s+a}{(s+a)^2+b^2}\right) = e^{-at}\cos bt$

3) $\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

4) $\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$

5) $\mathcal{L}^{-1}\left(\frac{b}{(s+a)^2+b^2}\right) = e^{-at}\sin bt$

6) $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$

7) $\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$

8) $\mathcal{L}^{-1}\left(\frac{1}{(s+a)^2}\right) = te^{-at}$

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CASE $\zeta = 1$

(II) RESPONSE OF CRITICALLY DAMPED for UNIT STEP i/p.

We know, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

as $r(t) = 1 \Rightarrow R(s) = \frac{1}{s}$.

$$\Rightarrow C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)}$$

$$\Rightarrow C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$= \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$\Rightarrow \omega_n^2 = A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs.$$

$\hookrightarrow s = 0$

$\Rightarrow \omega_n^2 = A\omega_n^2$

$\Rightarrow \boxed{A = 1}$

$\hookrightarrow s = -\omega_n$

$\Rightarrow \omega_n^2 = -C\omega_n$

$\Rightarrow \boxed{C = -\omega_n}$

$\hookrightarrow \omega_n^2 = (s+\omega_n)^2 + Bs(s+\omega_n) - s\omega_n$

$= s^2 + \omega_n^2 + 2s\omega_n - s\omega_n + Bs^2 + s\omega_n$

$= s^2 + \omega_n^2 + s\omega_n + B(s^2 + s\omega_n)$

for $B = (-1)$

$\Rightarrow \omega_n^2 = \omega_n^2 + (s^2 + s\omega_n) - 1(s^2 + s\omega_n)$

$\Rightarrow \boxed{B = -1}$ ✓

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So, we get

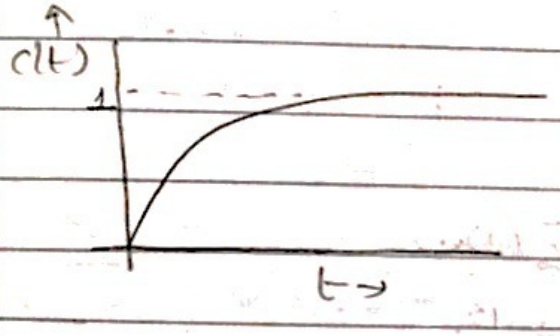
$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Now, taking $\mathcal{L}^{-1}$ both sides, we get

$$c(t) = 1 - e^{-\omega_n t} - \mathcal{L}^{-1}\left(\frac{\omega_n}{(s+\omega_n)^2}\right)$$

$$= 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}$$

$$\Rightarrow c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$



CASE (III) $0 < \zeta < 1$ RESPONSE OF UNDER DAMPED SYSTEM for UNIT STEP INPUT

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Rightarrow \omega_n^2 = A(s^2 + 2\zeta\omega_n s + \omega_n^2) + s(Bs + C)$$

$$\omega_n^2 = s^2(A+B) + s(2\zeta\omega_n A + C) + \omega_n^2(A)$$

equating coeff.

$$\Rightarrow A = 1,$$

$$B = -1 \quad (\because A + B = 0)$$

$$\Rightarrow C = -2\zeta\omega_n \quad (\because 2\zeta\omega_n A + C = 0)$$

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$$\Rightarrow C(s) = \frac{1}{s} + \frac{(-1)s + (-2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{s(s+2\zeta\omega_n) + \omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) - \zeta^2\omega_n^2 + \omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$= \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$\hookrightarrow \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n \times \omega_d}{((s+\zeta\omega_n)^2 + \omega_d^2)\omega_d}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n\omega_d}{((s+\zeta\omega_n)^2 + \omega_d^2)\omega_d}$$

$$= \left(\frac{s+a}{(s+a)^2+b^2}\right) - \frac{\zeta\omega_n}{\omega_d}\left(\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right)$$

$$\Rightarrow \left(\frac{b}{(s+a)^2+b^2}\right)$$

$$\Rightarrow c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\cdot e^{-\zeta\omega_n t}\sin\omega_d t$$

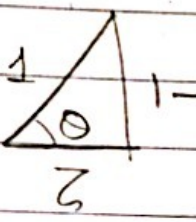
$$= 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\cdot\sin\omega_d t\right]$$

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$$\Rightarrow c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$= 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]$$



$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right)$$

$$\Rightarrow c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin(\omega_d t + \theta)\right)$$

*

- response oscillates with a frequency of ω_d.
- real parts of complex roots controls amplitude while imaginary part influences frequency of oscillns.

CASE (IV) $\zeta > 1$

RESPONSE of 2nd ORDER OVERDAMPED SYS. for UNIT STEP i/p.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Control Systems Notes - Akshansh

For the denominator of TF, as proved before, ∃ 2 roots of characteristic eqⁿ:-

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

Let $s_a = -s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

& $s_b = -s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

So, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s-s_1)(s-s_2)}$ or $\frac{\omega_n^2}{(s+s_a)(s+s_b)}$

$$\Rightarrow C(s) = \frac{1}{s}\cdot\frac{\omega_n^2}{(s+s_a)(s+s_b)} = \frac{A}{s} + \frac{B}{s+s_a} + \frac{C}{s+s_b}$$

$$\omega_n^2 = A(s+s_a)(s+s_b) + Bs(s+s_b) + Cs(s+s_a)$$

→ $s=0$.

$$\Rightarrow A = \frac{\omega_n^2}{s_a s_b} = \frac{\omega_n^2}{\zeta^2\omega_n^2 + \omega_n^2(\zeta^2-1)} = 1$$

→ $s = -s_a$

$$\Rightarrow B = \frac{\omega_n^2}{(-s_a)(s_b-s_a)} = \frac{-\omega_n}{2s_a\sqrt{\zeta^2-1}}$$

→ $s = -s_b$.

$$\Rightarrow C = \frac{-\omega_n^2}{s_b(s_a-s_b)} = \frac{\omega_n}{2s_b\sqrt{\zeta^2-1}}$$

$$\Rightarrow C(s) = \frac{\omega_n^2}{s s_a s_b} + \frac{\omega_n^2}{s(s+s_a)(s_b-s_a)} + \frac{\omega_n^2}{s(s+s_b)(s_a-s_b)}$$

or, $$C(s) = \frac{1}{s} - \frac{\omega_n}{2s_a(\sqrt{\zeta^2-1})(s+s_a)} + \frac{\omega_n}{2s_b\sqrt{\zeta^2-1}(s+s_b)}$$

Control Systems Notes - Akshansh

$$\Rightarrow C(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{1}{s_a(s+s_a)} - \frac{1}{s_b(s+s_b)}\right)$$

Applying $\mathcal{L}^{-1}$

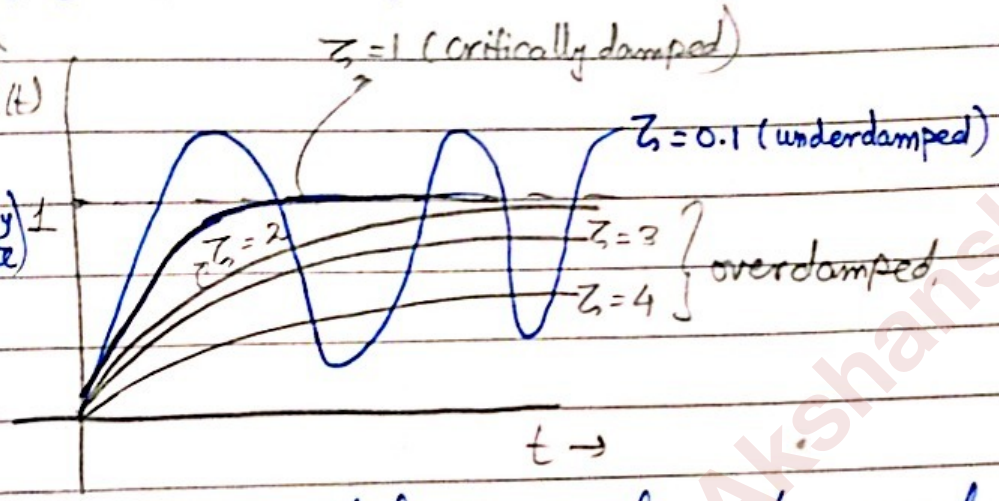
$$\Rightarrow c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{1}{s_a}e^{-s_a t} - \frac{1}{s_b}e^{-s_b t}\right]$$

$$\Rightarrow c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{e^{-(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t}}{\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}} - \frac{e^{-(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t}}{\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}\right]$$

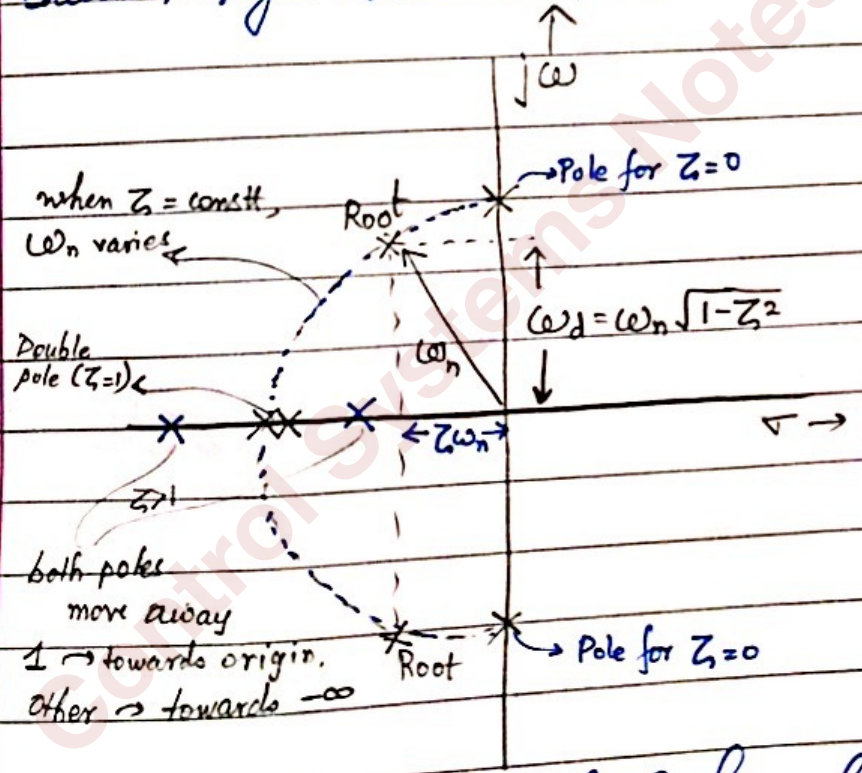
Control Systems Notes - Akshansh

* In most of real world cases, damping is slightly underdamped.

Response purely exponential - damping is so high that there are no oscill^{ns}.



* If ζ is more, c(t) takes more time to reach steady state; Sys. becomes slower.



S-plane

- for ζ = 0, poles lie on imaginary axis.
- for ζ = 1, poles lie on -ve real axis

* Rel^{n}ship b/w roots (closed loop poles) of characteristic eq^{n} of 2nd order sys. & ζ, ωn & ωd.

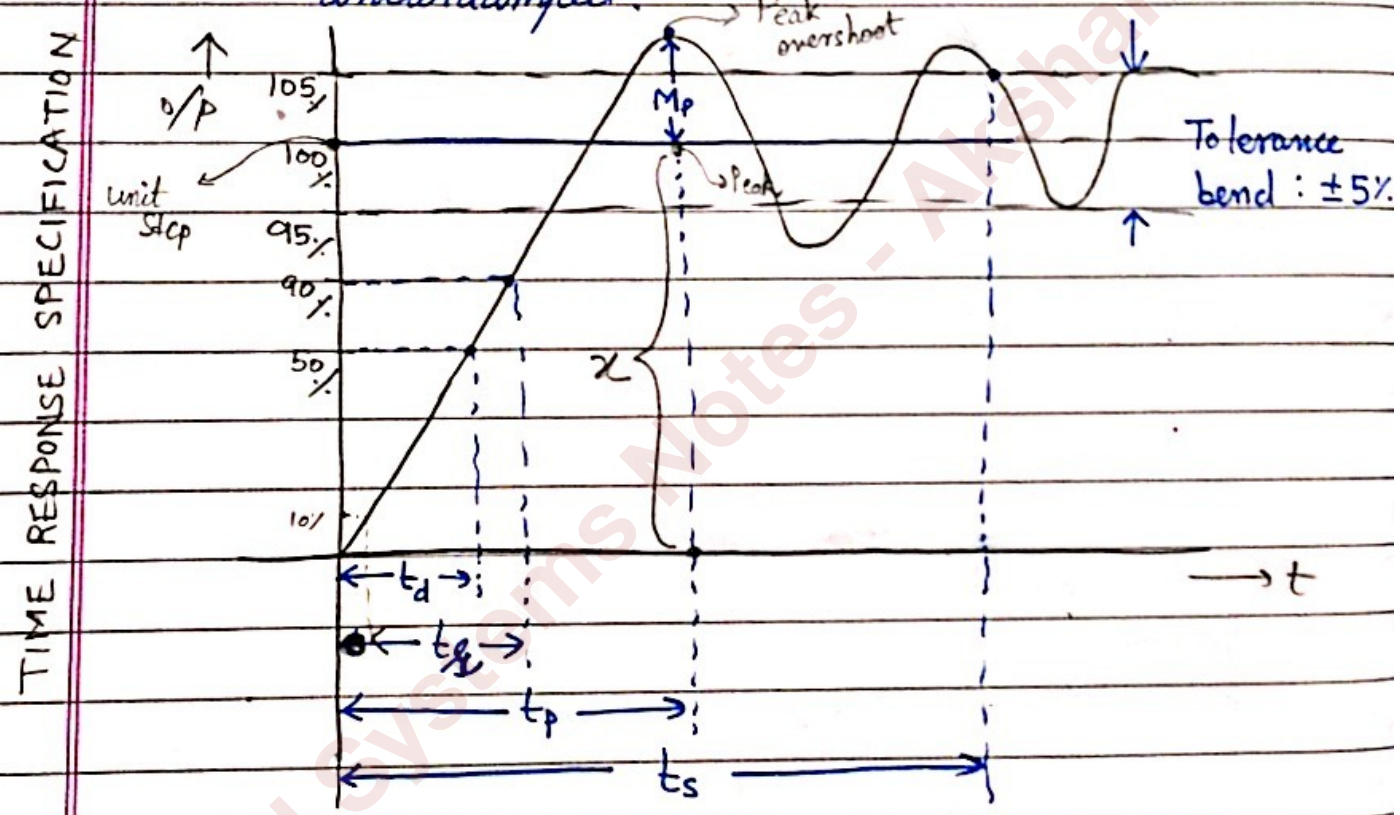
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* OLTF : Open loop Transfer fn
* CLTF : Closed loop Transfer fn

TIME RESPONSE SPECIFICⁿ

→ Seeing the specificⁿs, we can tell anything about a sys.

→ Time response of 2nd & higher order sys are normally designed with ζ ≤ 1, SLIGHTLY underdamped.



- t_d : delay time
  : Time reqd for response to reach 50% of final value in 1st attempt.
- t_r : rise time
  : Time reqd for response to rise from 10% to 90% of final value in the 1st attempt.

defin for overdamped sys

→ 10 - 90 % : overdamped sys
→ 0 - 100 % : underdamped
→ 5 - 95 % : critically damped

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- t_p : Peak time
  : time reqd for response to reach the peak, or, the peak overshoot in the 1st attempt.
- M_p : the normalised diff. b/w time response peak & steady state i/p.
  Peak overshoot.

$$\therefore M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100.$$

$c(t_p)$ → response at t_p; $c(\infty)$ → x_0 as in fig.

- t_s : Settling time
  : time reqd for response to reach & stay within a specified tolerance band. (2-5%) of its final value.
- e_{ss} : steady state error.

✱ Derivation of t_p

for underdamped sys., as derived before,

o/p or response $\rightarrow c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi)$

↳ time response specific^n for 2nd ord. sys.

Now, $\frac{d}{dt} c(t) = 0.$

$$\Rightarrow -\frac{e^{-\zeta\omega_n t}(-\zeta\omega_n)}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi) - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (\omega_d) \cos(\omega_d t + \phi)$$

$\omega_d \rightarrow \omega_n\sqrt{1-\zeta^2}$

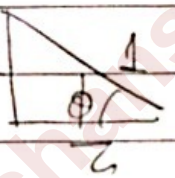
Control Systems Notes - Akshansh

$$\Rightarrow \frac{\zeta\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi) - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\omega_n\sqrt{1-\zeta^2}\right)\cos(\omega_d t+\phi) = 0$$

$$\Rightarrow \left[\zeta\sin(\omega_d t+\phi) - \sqrt{1-\zeta^2}\cos(\omega_d t+\phi)\right]\left(\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right) = 0$$

conversion

$$\Rightarrow \tan(\omega_d t+\phi) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$



$$\Rightarrow \text{Since, } \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \text{ where, } \tan(\omega_d t+\phi) = \tan\phi$$

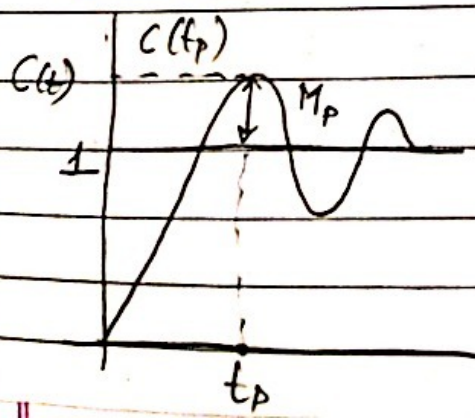
[From trigonometry, $\tan(n\pi+\phi) = \tan\phi$
If $n\pi = \omega_d t$, when $n = 1, 2, 3, \ldots$]

At $t = t_p$, (time req^d for first peak overshoot),
$n = 1$, $t = t_p$.

Hence, $$\boxed{t_p = \frac{n\pi}{\omega_d} = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \text{ sec}}$$

Imp.

☆ Derivation of M_p



From fig, $M_p = C(t_p) - 1$.

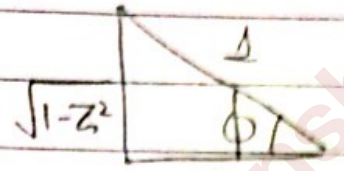
$$\Rightarrow M_p = \frac{-e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \times \sin(\omega_d t_p + \phi),$$

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Now, $\because t_p = \frac{\pi}{\omega_d}$,

$$M_p = -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\pi + \phi) \quad (\sin\phi \to -\sin\phi)$$

$$= \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\phi$$



$$= \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \times \sqrt{1-\zeta^2}$$

$$= +e^{-\zeta\omega_n t_p}$$

$$= e^{\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

$\Rightarrow M_p = e^{\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$, $\because t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

$\therefore \% M_p = 100\, e^{\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

☆ Derivation of t_r

$c(t_r) = 1$ for unit step i/p. → for underdamped sys.

$$= 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \phi)$$

↳ Only possible when $\underbrace{\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}}_{\neq 0} \underbrace{\sin(\omega_d t_r + \phi)}_{\text{Should be zero.}} = 0$,

$\Rightarrow \sin(\omega_d t_r + \phi) = 0. \Rightarrow \omega_d t_r + \phi = n\pi$, $n \in Z^+$

Control Systems Lectures - Akshansh

Here, first overshoot, $n = 1$,

$$\Rightarrow \omega_d t_r + \phi = \pi \; ; \quad \boxed{t_r = \frac{\pi - \phi}{\omega_d}}$$

✱ Derivation of t_s

t_s - time req^d by the o/p to settle down within $\pm 2\%$ of tolerance band. t_s is the time when o/p becomes 98% of final value & remains within the range of 2% as $t \to \infty$.

$$c(t)\Big|_{t=t_s} = 0.98 \text{ for unit i/p.}$$

At $t = t_s$, transient oscillatory terms completely vanishes. Here, the term $e^{-\zeta\omega_n t}$ only controls final value of o/p.

$$c(t)\Big|_{t=t_s} = 1 - e^{-\zeta\omega_n t_s} = 0.98.$$

$$\Rightarrow e^{-\zeta\omega_n t_s} = 0.02.$$

$$\Rightarrow -\zeta\omega_n t_s = \ln(0.02) = -3.912.$$

$$\Rightarrow t_s = \frac{-3.912}{-\zeta\omega_n} = \frac{3.912}{\zeta\omega_n} \simeq \frac{4}{\zeta\omega_n} \text{ for 2\% band}$$

for 5% tolerance band,

$$c(t)\Big|_{t=t_s} = 0.95.$$

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$= 1 - e^{-\zeta\omega_n t_s}$.

$\Rightarrow e^{-\zeta\omega_n t_s} = 0.05$

$\Rightarrow t_s = \frac{3}{\zeta\omega_n}$

In general, for a specified tolerance error band of e%,

$$t_s = \left| \frac{\ln(\%e)}{\zeta\omega_n} \right|$$

→ +ve value

* t_d : delay time $= \frac{1 + 0.7\zeta}{\omega_n}$

→ deriv^n not req^d.

Q. A sys. has OLTF as $G(s)H(s) = \frac{10}{s(s+5)}$

Find undamped natural frequency $= \omega_n$.

$-\zeta$

$-\omega_d$

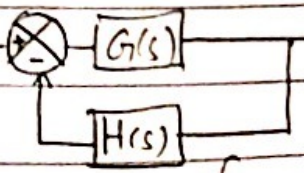
$-t_r$

$-t_p$

$-M_p$

$-t_s$: with 2% criterion

Idea :



$TF = \frac{G(s)}{1 + G(s)H(s)}$; for OLTF, f/b ✗.

$\Rightarrow \boxed{1 + G(s)H(s) = 0}$

Sol^n Given OLTF

$\Rightarrow 1 + G(s)H(s) = 0$. : Characteristic eq^n.

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☆ Note: for a TF = Numerator / Denominator

Num. = Gain ; Den = Characteristic eqⁿ

Puffin Date Page

$\Rightarrow 1 + \frac{10}{s(s+5)} = 0$

$\Rightarrow s^2 + 5s + 10 = 0$

$\left[s^2 + (2\zeta\omega_n)s + \omega_n^2 = 0 \right]$

Comparing, $\omega_n^2 = 10 \Rightarrow \omega_n = \sqrt{10} = 3.16$

& $2\zeta\omega_n = 5 \Rightarrow \zeta = \frac{5}{2\sqrt{10}} = 0.7905$

$\downarrow$

Underdamped sys. ($\zeta < 1$)

Now, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ ✓ rad/s $= 1.937$ rad/s

$t_r = \frac{\pi - \phi}{\omega_d}$ → $\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 1.2808$ s

$= 37.76° = 0.659^c$

$t_p = \frac{\pi}{\omega_d}$ ✓ $= 1.621$ s

$\%M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100 = 1.73\%$

$t_s = \frac{\ln(0.02)}{\zeta\omega_n} = |-1.56 s| = 1.56$ s (remove -ve)

Q A unity f/b sys. is characterised by an OLTF s.t., $G(s) = \frac{k}{s(s+16)}$ → ?

Determine k s.t. sys has a damping ratio of 0.5. For this value of k, find t_s for a 5% & 2% tolerance band, M_p, t_p.

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For OLTF ; $1 + G(s)H(s) = 0$

$\Rightarrow 1 + G(s) \cdot 1 = 0$

$\Rightarrow 1 + \frac{k}{s(s+10)} = 0$

$\Rightarrow s^2 + 10s + k = 0$

$\left(s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \right)$

Given,

$\zeta = 0.5$ (underdamped)

$\Rightarrow 2(0.5)\omega_n = 10$ } Comparing

$\Rightarrow \boxed{\omega_n = 10}$ }

$\Rightarrow k = \omega_n^2 = \underline{100}$ ✓

For :- (2%) & (5%)

$t_s = \frac{\ln(0.02)}{\zeta\omega_n}$ & $\frac{\ln(0.05)}{\zeta\omega_n}$

$= \frac{\ln(0.02)}{0.5 \times 10}$ & $\frac{\ln(0.05)}{5}$

$t_s = (+)0.782$ sec & $(+)0.599$ sec

→ removing -ve

$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100$

$\Rightarrow M_p = 16.30\%$

$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.36$ sec

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Q A sys is characterised by

$$\frac{d^2C}{dt^2} + 4\frac{dC}{dt} + 8C = 20r \quad \rightarrow (1)$$

where C(s) is o/p & R(s) is i/p. Evaluate all the time domain specifications & also the response for a step i/p of 2 units.

LT of eqn (1)

$$\Rightarrow s^2C(s) - sC(0) - C'(0) + 4[sC(s) - C(0)] + 8C(s) = 20R(s)$$

($C(0)$ and $C'(0)$ are 0 for TF)

$$\Rightarrow s^2C(s) + 4sC(s) + 8C(s) = 20R(s)$$

$$\Rightarrow (s^2 + 4s + 8)C(s) = 20R(s)$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{20}{s^2 + 4s + 8}$$

Given - step i/p = 2 units.

$$\Rightarrow r(t) = 2$$

$$\Rightarrow R(s) = \mathcal{L}(r(t)) = \frac{2}{s}$$

$$\Rightarrow C(s) = \frac{2}{s}\left[\frac{20}{s^2 + 4s + 8}\right]$$

$$\Rightarrow C(s) = \frac{40}{s(s^2 + 4s + 8)}$$

Now, $C(t) = \mathcal{L}^{-1}(C(s))$

↑ response

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$$\Rightarrow C(\omega) = \mathcal{L}^{-1}\left[\frac{40}{s(s^2+4s+8)}\right]$$

Idea, we want as :-

$$\mathcal{L}^{-1}\left(\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

$$\Rightarrow C_{(t)} = 5\,\mathcal{L}^{-1}\left[\frac{8}{s(s^2+4s+8)}\right]$$

$$\Rightarrow C_{(t)} = 5\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi)\right]$$

$C_{(t)}$ ← response

Std result

here, characteristic eqn :-

$$s^2+4s+8=0$$

$$s^2+2\zeta\omega_n s+\omega_n^2=0$$

$$\Rightarrow 2\zeta\omega_n = 4 \quad \& \quad \omega_n^2 = 8$$

$$\Rightarrow \zeta = \frac{1}{\sqrt{2}} \qquad \Rightarrow \omega_n = 2\sqrt{2}$$

$$\Rightarrow \omega_n = 2.828$$

$$\Rightarrow \zeta = 0.707$$

Now, t_s (not told the tolerance band. So, solve for both 2% & 5%)

$$t_s(2\%) = \left|\frac{\ln(0.02)}{\zeta\omega_n}\right| = 1.95 \text{ s}$$

$$t_s(5\%) = \left|\frac{\ln(0.05)}{\zeta\omega_n}\right| = 1.497 \text{ s}$$

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$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100$$

$$M_p = 4.321\ \%$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.57\ s$$

$$t_r = \frac{\pi - \phi}{\omega_d} = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega_d}$$

$$\Rightarrow t_r = 1.17\ s$$

$$t_d = \frac{1 + 0.7\zeta}{\omega_n} = 0.5286\ s$$

Now, substituting values in $c(t)$

$$c(t) = 5\left[1 - (1.414)e^{-2t}\sin\left(2t + \frac{\pi}{4}\right)\right]$$

Q $\dfrac{d^2c}{dt^2} + 7\dfrac{dc}{dt} + 12c = 12r \longrightarrow (1)$

Find response for unit step i/p

$$\Downarrow$$

$$r(t) = 1$$

$$\Rightarrow R(s) = \frac{1}{s}$$

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Finding $\mathcal{L}^{-1}$ of eqⁿ (1).

$\Rightarrow s^2 C(s) + 7s\,C(s) + 12\,C(s) = 12\,R(s)$

$\Rightarrow (s^2 + 7s + 12)\,C(s) = 12\,R(s)$

$\Rightarrow \frac{C(s)}{R(s)} = \frac{12}{s^2+7s+12}$

$\Rightarrow C(s) = \frac{1}{s}\left(\frac{12}{s^2+7s+12}\right)$

$C(s) = \frac{12}{s(s^2+7s+12)}$

$\Downarrow\Downarrow$

$\left(\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\right)$

Comparing, we get

$\Rightarrow C(t) = \mathcal{L}^{-1}(C(s)) = \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right]$

→ ∵ this is eqⁿ for underdamped. Find ζ first & check.

Characteristic eqⁿ :-

$s^2 + 7s + 12 = 0$

$2\zeta\omega_n = 7, \quad \omega_n^2 = 12$

$\Rightarrow \zeta = 1.0103$ & $\omega_n = 3.46$.

→ $\zeta > 1$

→ overdamped

(191) : Use the overdamped eqⁿ as solved before.

Control Systems Notes - Akshansh

(M2) Now, $C(t) = \mathcal{L}^{-1}\left[\frac{12}{s(s^2+7s+12)}\right]$

Finding by partial fractions

$\therefore \mathcal{L}^{-1}\left[\frac{12}{s(s+3)(s+4)}\right]$

$= \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+4}\right]$

$\Rightarrow 12 = A(s+3)(s+4) + B(s)(s+4) + Cs(s+3)$

$\Rightarrow B = -4$

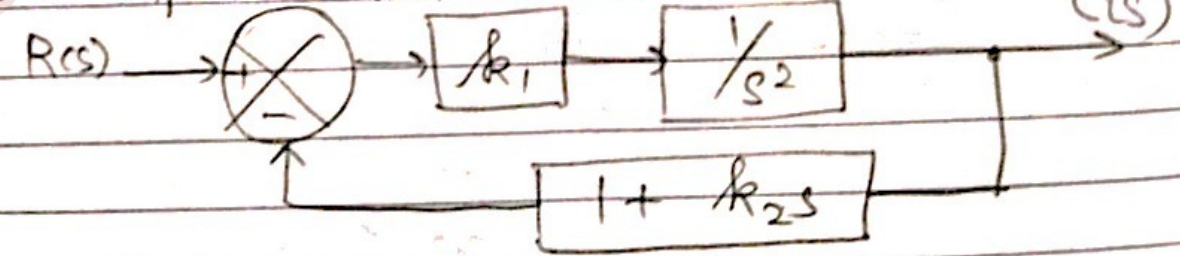
$C = 3$

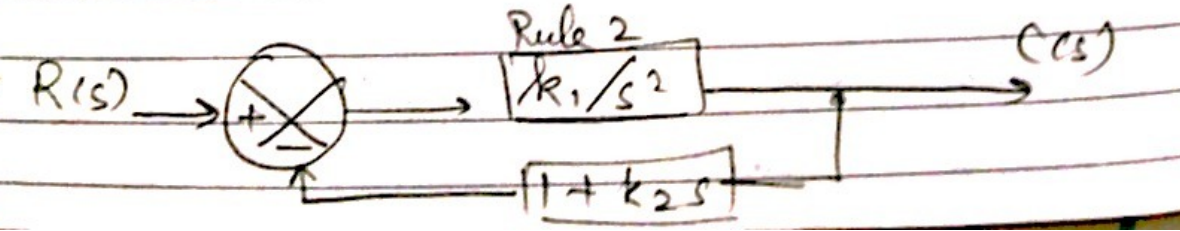
$A = 1$

$\Rightarrow c(t) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{4}{s+3} + \frac{3}{s+4}\right]$

$\Rightarrow c(t) = 1 - 4e^{-3t} + 3e^{-4t}$ Ans

Q For a control sys shown, find the gains k_1 & k_2 s.t $M_p = 25\%$, $t_p = 4s$. Assume unit step i/p





Control Systems Notes - Akshansh

Rule 9

$$R(s) \rightarrow \boxed{\frac{k_1/s^2}{1+(k_1/s^2)(1+k_2 s)}} \rightarrow C(s)$$

Here, $G_1(s) = \frac{k_1}{s^2}$, $H(s) = 1 + k_2 s$

Now, CLTF $= \dfrac{k_1/s^2}{1+\left(\frac{k_1}{s^2}\right)(1+k_2 s)}$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{k_1}{s^2 + k_1 + k_1 k_2 s}$$

$$\Rightarrow C(s) = \frac{1}{s}\left(\frac{k_1}{s^2 + (k_1 k_2)s + k_1}\right)$$

$$\Rightarrow \left(\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}\right)$$

Comparing

$\Rightarrow$ Characteristic eqⁿ

$= s^2 + k_1 k_2 s + k_1$

$2\zeta\omega_n = k_1 k_2$ & $\omega_n^2 = k_1$

$\Rightarrow \zeta = \dfrac{\sqrt{k_1}\, k_2}{2}$ — (B)

Now, $t_p = 4 = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

$$\Rightarrow 4 = \frac{\pi}{\sqrt{k_1}\sqrt{1-\frac{k_1 k_2^2}{4}}}$$

Control Systems Notes - Akshansh

$$\Rightarrow k_1(4 - k_1 k_2^2) = \frac{\pi^2}{4} \longrightarrow (1)$$

Also, $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = .25$

$$\Rightarrow \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} = \ln(.25)$$

$$\Rightarrow \frac{-\zeta}{\sqrt{1-\zeta^2}} = \left[\frac{\ln(.25)}{\pi}\right]$$

$$\Rightarrow \frac{\zeta^2}{1-\zeta^2} = \left[\frac{\ln(.25)}{\pi}\right]^2 \longrightarrow (A)$$

$$\Rightarrow \left(\frac{k_1 k_2^2}{4}\right)\left[\frac{4}{4 - k_1 k_2^2}\right] = \left[\frac{\ln(.25)}{\pi}\right]^2$$

$$\Rightarrow \frac{k_1 k_2^2}{4 - k_1 k_2^2} = \frac{1.9218}{\pi^2} \longrightarrow (2)$$

From (A)

$$\frac{\zeta^2}{1-\zeta^2} = 0.1947$$

$$\Rightarrow \zeta^2(1 + 0.1947) = 0.1947$$

$$\Rightarrow \zeta = \sqrt{\frac{0.1947}{1.1947}} = 0.4036$$

from (B) & (1).

$$k_1(4 - 4\zeta^2) = \frac{\pi^2}{4} \Rightarrow k_1 = 0.73688.$$

From (B), $k_2 = \frac{2\zeta}{\sqrt{k_1}} = 0.94033$

Ans

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Q. The OLTF of a unity feedback sys. is :-

$G(s) = \frac{k}{(1+Ts)^2}$, where k, T: consts.

Determine the factor by which the gain, k should be multiplied s.t. the peak overshoot, M_p of unit step response be reduced from 75% to 25%

Idea :- When gain, k : $M_p = 75\%$ } 2 eqns, 2 variables
gain, $(2)k$: $M_p = 25\%$ } ✓
↳ const

Note :- M_p is diff^t. $\Rightarrow \zeta$ will vary.

Now, $G(s) = G(s)H(s) = G(s)(1) = \frac{k}{(1+Ts)(s)}$

$G(s) = \frac{k}{Ts^2+s}$

Now, CLTF $= \frac{G(s)}{1+G(s)} = \frac{\left(\frac{k}{Ts^2+s}\right)}{1+\left(\frac{k}{Ts^2+s}\right)}$

$\Rightarrow \frac{C(s)}{R(s)} = \frac{k}{Ts^2+s+k}$

$\Rightarrow C(s) = \frac{1}{s}\left(\frac{k}{Ts^2+s+k}\right)$

$= \frac{k/T}{s\left(s^2+\left(\frac{1}{T}\right)s+\left(\frac{k}{T}\right)\right)}$

$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$

Control Systems Notes - Akshansh

Comparing, we get

$$\frac{k}{T} = \omega_n^2 \quad \& \quad 2\zeta\omega_n = \frac{1}{T} \longrightarrow (A)$$

Now,

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = .75$$

$$\Rightarrow \frac{-\pi\zeta_1}{\sqrt{1-\zeta_1^2}} = \ln(.75)$$

$$\Rightarrow \frac{\zeta_1^2}{1-\zeta_1^2} = \left(\frac{\ln(0.75)}{\pi}\right)^2 \Rightarrow \boxed{\zeta_1 = 0.0912}$$

From (A).

$$\omega_n = \sqrt{\frac{k}{T}} \Rightarrow 2\zeta\sqrt{\frac{k}{T}} = \frac{1}{T}$$

$$\Rightarrow \zeta = \frac{1}{2\sqrt{kT}} \longrightarrow (2)$$

Notation, for $M_p = 0.75$, $\zeta = \zeta_1 = 1/2\sqrt{kT}$

$M_p = 0.25$ $\quad \zeta = \zeta_2 = 1/2\sqrt{kT}$

Now, $(M_p)_{new} = e^{\frac{-\pi\zeta_2}{\sqrt{1-\zeta_2^2}}} = 25$

$$\Rightarrow \frac{-\pi\zeta_2}{\sqrt{1-\zeta_2^2}} = \ln(25)$$

$$\Rightarrow \frac{\zeta_2^2}{1-\zeta_2^2} = \left(\frac{\ln(25)}{\pi}\right)^2$$

$$\Rightarrow \zeta_2 = 0.4036.$$

Control Systems Notes - Akshansh

Now, $\frac{1}{2\sqrt{kT}} = 0.0912$

& $\frac{1}{2\sqrt{k'T}} = 0.4036$

$\Rightarrow \sqrt{\frac{k'}{k}} = \frac{0.0912}{0.4036}$

$\Rightarrow k' = k\left[\frac{0.0912}{0.4036}\right]^2$

$\Rightarrow k' = (0.05510)\,k$

The factor ✓

* The 2 roots of the characteristic eqⁿ, $s^2 + 2\zeta\omega_n s + \omega_n^2$ are also called Closed loop poles (CL poles)

Q. A CL Sys. has 2 complex conjugate poles, $s = -2 \pm j(1)$

Find all specificⁿˢ assuming Std. 2nd order unit step fⁿ ✓

Given :- $s = -2 \pm j$

$\Rightarrow$ 2 poles are $(s+2-j)$ & $(s+2+j)$

So, characteristic eqⁿ is

$$(s+2-j)(s+2+j) = 0$$

$$\Rightarrow s^2 + (2-j+2+j)s + (2-j)(2+j) = 0$$

Control Systems Notes - Akshansh

$\Rightarrow s^2 + 4s + 5 = 0$

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$\Rightarrow$

$\omega_n^2 = 5 \Rightarrow \omega_n = \sqrt{5}$

$2\zeta\omega_n = 4 \Rightarrow \zeta\omega_n = 2 \Rightarrow \zeta = \frac{2}{\sqrt{5}}$

Now, All specifications:

(a) $t_s(2\%) = \left|\frac{\ln(0.02)}{\zeta\omega_n}\right|$, $t_s(5\%) = \left|\frac{\ln(0.05)}{\zeta\omega_n}\right|$

$t_s(2\%) = 1.956$ sec, $t_s(5\%) = 1.497$ sec.

(b) $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100$

$\Rightarrow M_p = 0.186\ \%$

(c) $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 3.14$ sec.

(d) $t_{s_2} = \frac{\pi - \phi}{\omega_d} = \pi - \frac{\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega_d}$

$= \pi - \frac{\tan^{-1}(1/2)}{\omega_n\sqrt{1-\zeta^2}} = 2.6779$ sec

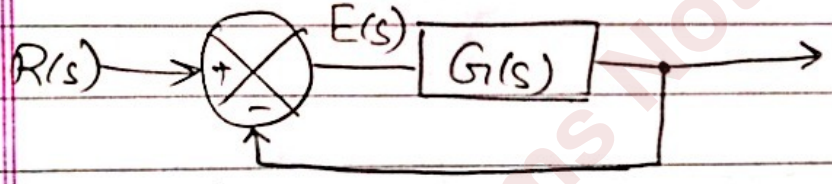
(e) $t_d = \frac{1 + 0.7\zeta}{\omega_n} = 0.7272$ sec

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✡ STEADY STATE ERRORS & CONSTANT

- An extremely imp. aspect of sys. performance.
- A measure of sys. accuracy
- errors arise from nature of i/p, type of sys. & non linearities of sys. components such as static friction, backlash etc.
- errors are aggravated by amplifier drifts, aging or deterioration.
- Steady state performance of a stable CS is judged by e_{ss} to step, ramp & parabolic i/p.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\Rightarrow C(s) = R(s) \times \frac{G(s)}{1+G(s)}$$

* Final value theorem,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \frac{s \cdot R(s)}{1+G(s)}$$

→ e_{ss} depends on i/p $R(s)$ & forward path gain $G(s)$

Also,

$$C(s) = E(s)\,G(s)$$

$$\text{or } E(s) = \frac{C(s)}{G(s)}$$

$$\mathcal{L}(e(t)) = \frac{R(s) \times G(s)}{[1+G(s)]\,G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

Control Systems Notes - Akshansh

1. e_{ss} due to unit step i/p (position i/p)

$\hookrightarrow r(t) = 1 \; (= u(t))$

$\Rightarrow R(s) = \frac{1}{s}$

So,

$$e_{ss} = \lim_{s \to 0} \frac{s \cdot R(s)}{1 + G_1(s)} = \lim_{s \to 0} \frac{s \cdot \frac{1}{s}}{1 + G_1(s)} = \frac{1}{1 + G_1(0)}$$

$$\Rightarrow \boxed{e_{ss}\Big|_{\text{unit step i/p}} = \frac{1}{1 + G_1(0)} = \frac{1}{1 + K_p}}$$

$\hookrightarrow$ $K_p = G_1(0) =$ Position error constt

2. e_{ss} due to unit ramp i/p (velocity i/p)

$\hookrightarrow r(t) = t$

$\Rightarrow R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s \to 0} \frac{s \cdot \frac{1}{s^2}}{1 + G_1(s)} = \lim_{s \to 0} \frac{1}{sG_1(s) + s} = \lim_{s \to 0} \frac{1}{sG_1(s)}$$

$$\Rightarrow \boxed{e_{ss}\Big|_{\text{unit ramp i/p}} = \lim_{s \to 0} \frac{1}{sG_1(s)} = \frac{1}{K_v}}$$

$\hookrightarrow K_v = \lim_{s \to 0} s(G_1(s)) =$ velocity error constt.

3. e_{ss} due to parabolic i/p (Accl^n i/p)

$\hookrightarrow r(t) = \frac{t^2}{2}$

$\Rightarrow R(s) = \frac{1}{s^3}$ $\rightarrow$ $\left[\mathcal{L}\left(\frac{t^2}{2}\right) = \frac{1}{2} \cdot \frac{2!}{s^{2+1}}\right]$

Control Systems Notes - Akshansh

$$\therefore e_{ss} = \lim_{s\to 0} \frac{s\left(\frac{1}{s^3}\right)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2 G(s)} = \frac{1}{K_a}$$

$$\Rightarrow \boxed{e_{ss}\Big|_{\text{parabolic i/p}} = \lim_{s\to 0} \frac{1}{s^2 G(s)} = \frac{1}{K_a}}$$

$\hookrightarrow K_a = \lim_{s\to 0} s^2 G(s)$: accl^n error constt.

- The coeff.$_s$ can be found only if sys. is stable

* TYPE : A property of OLTF. : $G(s)H(s)$
ORDER : " " " CLTF : $\frac{G(s)}{1 \pm G(s)H(s)}$

* Types of feedback CS.

written in 2 std. forms

(i) Time constt form. (ii) Pole-zero form.

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots}{s^j (1+T_a s)(1+T_b s)\cdots}$$

eg: $\frac{40(s+1)}{s^1(s+1)(s+3)}$

$\hookrightarrow$ K : resultant sys. gain.

j : (type) of sys

$\hookrightarrow$ no. of poles at origin of OLTF, $G(s)H(s)$ of the sys.

$\hookrightarrow$ j = 0 : type 0
j = 1 : type 1
j = n : type n.

Note : Any term with "s" for OLTF always has
" $1 + (sx)$ " : Time constt.
" $() + (sx())$ " : Pole zero.

i.e, $\frac{K(3+s)}{s(1+s)(2+s)} \equiv \frac{3x\,k\left(1+\frac{s}{3}\right)}{2x\,s(1+s)\left(1+\frac{s}{2}\right)}$

Pole zero Time constant

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★ CLASS-I : STEP INPUT

1) Assuming : Sys is type 0.

$$\Rightarrow G(s)H(s) = \frac{k(1+T_1 s)(1+T_2 s)\cdots}{s^0(1+T_a s)(1+T_b s)\cdots}$$

$s^0 = 1$

For step i/p, $K_p = \lim_{s\to 0} G(s)H(s) = k$

$$e_{ss} = \frac{A}{1+K_p} = \frac{A}{1+k}$$ (for unit step i/p, $A = 1$)

2) Sys. is type 1

$$G(s)H(s) = \frac{K(1+T_1 s)(1+K_2 s)\cdots}{s(1+T_a s)(1+T_b s)\cdots}$$

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{1}{s}[\quad] \to \infty$$

$$\Rightarrow e_{ss} = \frac{A}{1+K_p} = 0$$

3) Type 2

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots}{s^2(1+T_a s)(1+T_b s)\cdots}$$

$$K_p = \lim_{s\to 0} G(s)H(s) \to \infty$$

$$\Rightarrow e_{ss} = \frac{A}{1+K_p} = 0$$

★ Inference: For any sys. of type > 0, $k_p \to \infty$ & $e_{ss} = 0$

(practically, a small error is present)

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☆ CLASS II : RAMP INPUT.

1) System is type 0.

$$G(s)H(s) = \frac{k(1+T_1 s)(1+T_2 s)\cdots}{s^0(1+T_a s)(1+T_b s)\cdots}$$

For ramp i/p, $K_v = \lim_{s\to 0} s\,G(s)H(s)$

$$= \lim_{s\to 0} \frac{s(k)(1+T_1 s)(1+T_2 s)\cdots}{(1+T_a s)(1+T_b s)\cdots}$$

$$= 0$$

$$e_{ss} = \frac{A}{K_v} \to \infty,$$

Inference : Type 0 sys will not follow ramp i/p of any magnitude & large e_{ss}.

2) Type 1

$$K_v = \lim_{s\to 0} sG(s)H(s) = K.$$

$$\Rightarrow e_{ss} = \frac{A}{K_v} = \frac{A}{K}.$$

↳ finite error for type 1 ramp i/p.

3) Type 2)

$$K_v = \lim_{s\to 0} s.G(s)H(s) = \lim_{s\to 0} s\left[\frac{k(1+T_1 s)\cdots}{s^2(1+T_a s)\cdots}\right]$$

$$\to \infty.$$

$$e_{ss} = \frac{A}{K_v} \to 0. \text{ (practically small)}$$

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* So, for any sys. of ~~unit~~ ramp i/p, with type ≥ 2, $K_v \to \infty$ & $e_{ss} \to 0$.

* CLASS - III : PARABOLIC INPUT

1) Type 0.

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots}{1(1+T_a s)(1+T_b s)\cdots}$$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = 0$$

$$e_{ss} = \frac{A}{K_a} \to \infty.$$

2) Type 1

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots}{s(1+T_a s)(1+T_b s)\cdots}$$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = 0$$

$$e_{ss} = \frac{A}{K_a} \to \infty.$$

3) Type 2

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = K$$

$$e_{ss} = \frac{A}{K_a} = \frac{A}{K} = \text{finite}.$$

4) Type 3

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) \to \infty,$$

$$e_{ss} \to 0$$

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* Inference : ① for unit parabolic i/p. & type ≥ 3, $K_a \to \infty$ & $e_{ss} \to 0$.

② Parabolic i/p causes error to be large in type 0 & 1 sys. Hence, it shouldn't be used as reference excite type 0 & type 1 sys.

* SUMMARY

Type of Sys.	Error coeff			e_{ss}		
	K_p	K_v	K_a	Step	Ramp	Parabolic
0	K	0	0	A/1+K	$\to \infty$	$\to \infty$
1	$\to \infty$	K	0	0	A/K	$\to \infty$
2	$\to \infty$	$\to \infty$	K	0	$\to 0$	0

Q. An OLTF is defined as

$$G(s)H(s) = \frac{10(s+2)(s+3)}{s(s+1)(s+5)(s+4)}$$

① Find all error coeff.
② Find error for a ramp i/p of magnitude 4.
③ Find error for an i/p $r(t) = 2 + t + t^2$
④ Which type of i/p gives a const. steady state ?
⑤ Find error by final value theorem for $r(t)$ given in part ③.

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* Ramp i/p gives a constt steady state error.

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① Here, $G(s)H(s) = \frac{10(s+2)(s+3)}{s(s+1)(s+5)(s+4)}$ → (A)

Zeroes → (A); Poles → $s(s+1)(s+5)(s+4)$

$= s^1 \Rightarrow$ Type 1.

So, $K_p = \lim_{s\to 0} G(s)H(s) = \to \infty$

$K_v = \lim_{s\to 0} sG(s)H(s) = 3$ $\left[\frac{s\,10(0+2)(0+3)}{s(0+1)(0+5)(0+4)}\right]$

$K_a = \lim_{s\to 0} s^2 G(s)H(s) = 0.$

② Ramp i/p of magnitude 4.

$\Rightarrow e_{ss} = \frac{4}{K_v} = \frac{4}{3}$ (from ①)

③ Given $r(t)$ = Input $= 3 + t + t^2$

$\equiv \underbrace{A_1 u(t)}_{\text{step i/p}} + \underbrace{A_2 t}_{\text{ramp i/p}} + \underbrace{A_3 \left(\frac{t^2}{2}\right)}_{\text{parabolic i/p}}$

$= 3(1) + 1(t) + 2\left(\frac{t^2}{2}\right)$

Note: i/p is a sum of step, ramp & parabolic i/p, e_{ss} will also be a fn of all the 3.

e_{ss1} (step i/p) $= \frac{A_1}{1+K_p} = \frac{3}{1+(\to\infty)} \to 0$

e_{ss2} (ramp i/p) $= \frac{A_2}{K_v} = \frac{1}{3}$: constt steady state error

e_{ss3} (parabolic i/p) $= \frac{A_3}{K_a} \to \infty$

So, $e_{ss(net)} = e_{ss1} + e_{ss2} + e_{ss3} = 0 + \frac{1}{3} + \underset{\to\infty}{(\to\infty)}$

$\to \infty$

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(5) By final value thm,

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$= \lim_{s \to 0} s\,E(s)$$

$$\Rightarrow e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1 + G(s)H(s)} \rightarrow (1)$$

By LT on $r(t)$, we get

$$R(s) = \mathcal{L}[3 + t + t^2]$$

$$= \frac{3}{s} + \frac{1!}{s^{1+1}} + \frac{2!}{s^{2+1}}$$

$$\Rightarrow R(s) = \frac{3}{s} + \frac{1}{s^2} + \frac{2}{s^3}$$

Now, using (1) & (A).

$$\Rightarrow e_{ss} = \lim_{s \to 0} \frac{s\left(\frac{3}{s} + \frac{1}{s^2} + \frac{2}{s^3}\right)}{1 + \frac{10(s+2)(s+3)}{s(s+1)(s+4)(s+5)}}$$

$$= \lim_{s \to 0} \frac{(3s^3 + s^2 + 2s)(s)(s+1)(s+4)(s+5)}{[s(s+1)(s+4)(s+5) + 10(s+2)(s+3)]\,s^3}$$

$$\Rightarrow e_{ss} \to \infty$$

Hence, verified.

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Q. For a sys., $G(s)H(s) = \frac{k}{s^2(s+2)(s+3)}$

(Pole-zero form)

→ (Type (2))

(1) Write above in time const. form.

$$\frac{k}{2\times 3\, s^2\left(1+\frac{s}{2}\right)\left(1+\frac{s}{3}\right)}$$

(2) Find the value of k to limit the steady state error to 10 when i/p to the sys is $r(t) = 1+10t+20t^2$

$r(t) = 1+10t+20t^2$

$\Rightarrow R(s) = \mathcal{L}(1+10t+20t^2)$

$= \frac{1}{s} + \frac{10}{s^2} + \frac{20\times 2}{s^3}$

$\Rightarrow R(s) = \frac{1}{s} + \frac{10}{s^2} + \frac{40}{s^3}$

or $r(t) = 1(1) + 10(t) + 40\left(\frac{t^2}{2}\right)$

$\Rightarrow r(t) \equiv A_1 u(t) + A_2 t + A_3\left(\frac{t^2}{2}\right)$

For type 2, $K_p = \infty$

$K_v = \infty$

$K_a = \lim_{s\to 0} \frac{s^2(k)}{s^3(s+2)(s+3)} = \frac{k}{6}$

$\star e_{ss(step)} = \frac{A_1}{1+K_p} = 0$ $\star e_{ss(ramp)} = \frac{A_2}{K_v} = 0$

$\star e_{ss(parabolic)} = \frac{A_3}{K_a} = \frac{40}{(K/6)}$

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$e_{ss\,(total)} = 10 = e_{ss\,step + ramp + parabolic}$

$\Rightarrow 10 = 0 + 0 + \underset{(K/6)}{40}$

$\Rightarrow 10(K/6) = 40$

$\Rightarrow \frac{K}{6} = 4$

$\Rightarrow K = 24$ Ans

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Chapter : Stability

↳ the ability of the sys. in such a way that small changes in sys. ~~o/p stroke~~ i/p / sys. parameters / initial cond^ns do not result in large change in system o/p.

→ In a stable sys., the response / o/p is predictable, finite & hence, stable for a given i/p.

→ BIBO Stability : Bounded Input Bounded Output Stability.
↳ A Linear Time Invariant sys is sys. if following 2 cond^ns are satisfied :

1. For any bounded (finite) i/p, the o/p is bounded (finite)
2. In the absence of i/p, the o/p tends towards zero or its equilibrium state, irrespective of its initial cond^ns.

↳ also called ASYMPTOTIC Stability

✱ Checking for stability :-
The TF has zero's (in numerator) & poles in denominator. They are plotted on the s-plane.

- If all roots in left half of s-plane : STABLE
- Even if one root is on imaginary or right half of s-plane : UNSTABLE
- ~~All~~ roots, NON-repeated on imaginary axis (jω axis) : LIMITEDLY or MARGINALLY STABLE

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Consider the eq^n :-

$$a_0 s^2 + a_1 s + a_2 = 0$$

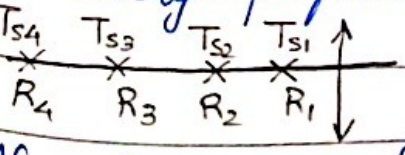
- On the basis of coefficient
  - If coeff. are +ve & no coeff is zero, then all the roots are in the left half of s-plane & sys is STABLE
  - If any coeff. is zero, then, some roots maybe on jω axis (imaginary axis) or right-half of s-plane & sys is UNSTABLE
  - If any coeff. is -ve, sys. is unstable.
- Nature of roots
  - If all roots of characteristic eq^n have -ve real parts, sys is STABLE
  - If any root has a +ve real part or ∃ a repeated root on jω axis, sys is UNSTABLE
  - If any roots ~~are~~ (non repeated) are present on jω axis, sys. is LIMITEDLY or MARGINALLY STABLE

★ RELATIVE Stability :

Stability that is relative to t_s (settling time).

If the settling time is less, the transience dies out faster & the sys. becomes more stable.

- The t_s of the pair of complex conjugate pairs is inversely proportional to real parts of roots $\left(t_s \propto \frac{1}{R}\right)$



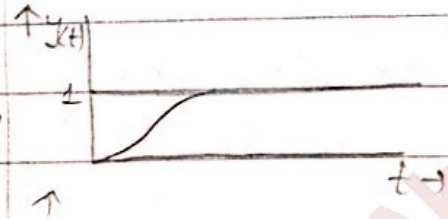
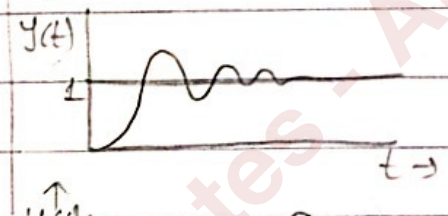
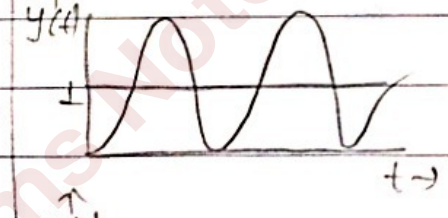
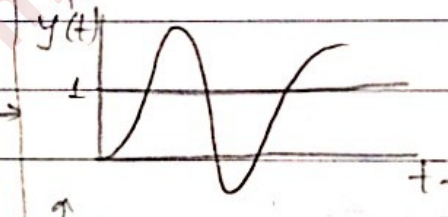
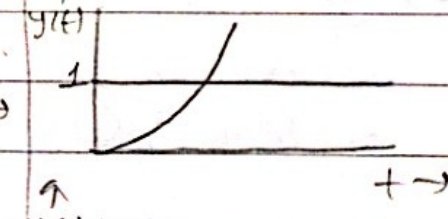
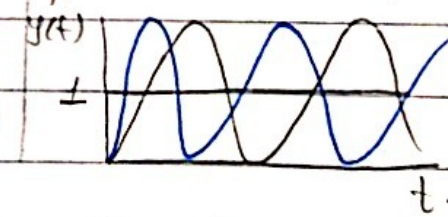
: here, on -ve axis, T_{s1} is nearest (max in value) & T_{s4} is farthest (min. in value)

- If roots (or pair of roots) move farther from imaginary axis: STABILITY ↑↑

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× : denote poles

⊙ : denote zero's

Location of CL poles in s-plane	Step Response	Stability Condⁿ
1. real, -ve, unequal; $\zeta > 1$		Absolutely stable
2. roots are real, equal, -ve; $\zeta = 1$		Absolutely stable
3. Complex conj. pair of roots; $0 < \zeta < 1$		Absolutely stable
4. non repeated roots; $\zeta = 0$		Marginally or Critically stable
5. imaginary, +ve roots; $0 > \zeta > -1$		Unstable
6. +ve real roots; $\zeta < -1$		Unstable
7. repeated roots; Repeated roots on jω axis → ⇒ 2 roots on same place		Unstable.

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ROUTH-HURWITZ CRITERION

Determining loc^n of poles w.r.t left half (or) right half of s plane without solving the characteristic eq^n.

$$CLTF,\ \frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n} = \frac{B(s)}{F(s)}$$

$F(s) = 0 \longrightarrow$ to find poles

$B(s) = 0 \longrightarrow$ to find roots.

→ for no root in Right half of s-plane.

⇓

Stable.

necessary cond^n { ✓ all coeff. of polynomial should have same sign.

✓ none of the coeff. vanishes, i.e, all powers of s must be present.

→ for all roots of characteristic eq^n in left half of s-plane,

Sufficient cond^n { ✓ all subdeterminants ~~must~~ must be +ve.

✡ Method of forming Hurwitz criterion:

for, $F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0$

$$H = \begin{vmatrix} a_1 & a_3 & a_5 & \dots & a_{2n-1} \\ a_0 & a_2 & a_4 & \dots & a_{2n-2} \\ 0 & a_1 & a_3 & \dots & a_{2n-3} \\ 0 & 0 & a_0 & \dots & a_{2n-4} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & & a_n \end{vmatrix}$$

(odd row: $a_1\ a_3\ a_5 \dots a_{2n-1}$; even row: $a_0\ a_2\ a_4 \dots a_{2n-2}$)

$D_1 = |a_1|$ → should be +ve

$D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$ → should be +ve

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$$D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$ → should be +ve.

Why other subdeterminants All should be +ve

* Determining stability by Hurwitz method:

eg

Let $F(s) = s^3 + s^2 + s + 4$.

↳ $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 4$

$n = 3$

So, $$H = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

→ $D_1 = |1| = 1$

→ $D_2 = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3$ — ve.

→ $D_3 = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = |H| = -12$

∴ D_2 & D_3 are -ve

So, Sys is unstable

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DISADVANTAGES of HURWITZ method:

- for higher order sys (n is large), lot of determinants to solve ⇒ time consuming.
- From the characteristic eqⁿ, we cannot say which roots are there & where do they lie. We can only talk about stability
- Difficult to predict marginally stable sys.

→ Disadvantages were overcome by Routh

⇓

ROUTH HURMITZ METHOD

Routh suggested a method of tabulating coeff. of characteristic eqⁿ in a particular way

→ ROUTH ~~WAY~~ ARRAY

If $F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$.

start with s^n coeff	even terms.
coeff. of s^{n-1}	odd terms
coeff of s^{n-2}	even terms
⋮	⋮
coeff of s^0	a_n

: way of writing

(PTO)

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s^n	a_0 (11)	a_2 (12)	a_4 (13)	a_6 - -	→ even only
s^{n-1}	a_1 (21)	a_3 (22)	a_5 (23)	a_7 - -	→ odd only
s^{n-2}	b_1	b_2	b_3 - - -		
s^{n-3}	c_1	c_2	c_3 - - -		
⋮					
s^0	a_n - - - - -				

Now,
writing the terms :-
b_1, b_2, b_3 - - - -
& c_1, c_2, c_3 - - - -

Idea :- for finding $b_1, b_2, \ldots$ - - -
keep $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ as it is and do →

for (b_1) → $\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \times (-1)$ / a_1

Note :- for b terms, always divide by a_1

(b_2) → $\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} \times (-1)$ / a_1 - - - - - -

Similar method continues for finding lower row terms.
Note :- Divide by $(\ )_{21}$ terms.

for finding $c_1, c_2, \ldots$ - - -
keep $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ constant & do →

(c_1) → $\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} \times (-1)$ / b_1

Note :- for c terms, always divide by b_1.

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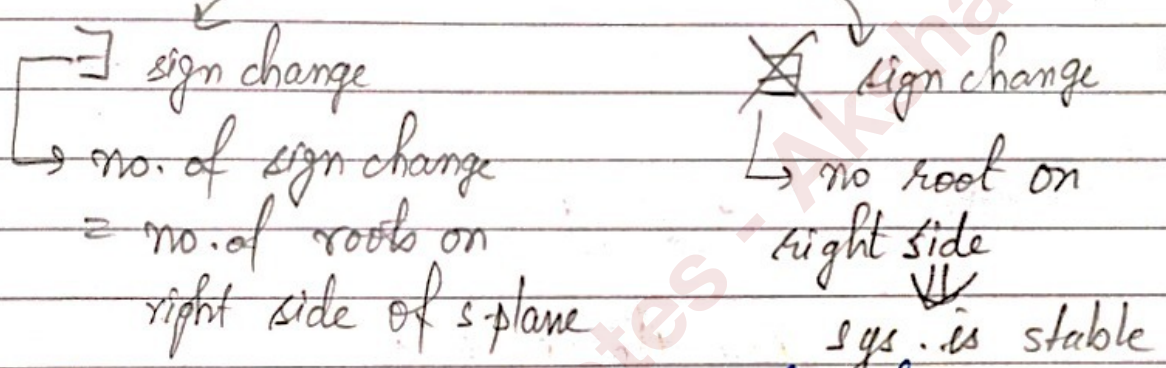
* ROUTH Criterion

↳ Idea for finding / checking for stability

↓

See the first column after finding values of $b_1, \ldots c_1 \ldots$ in Routh array.

- See for SIGN CHANGE



* Note :- When any element in the first column of the Routh's array becomes (zero), we cant say about it being +ve or -ve. In that case, this method FAILS.

eg ① : for $F(s) = s^3 + 6s^2 + 11s + 6 = 0$

	a_0	a_1
s^3	1	11
s^2	6	6
s^1	$\frac{66-6}{6}$. ( )	
s^0	$\frac{60-0}{10}$. ( )	

Array

; ∃ no change in sign of 1st column. So, stable ✓

3
ϵ
−∞

eg ② :- (Self) :- Consider $s^3 + 4s^2 + s + 16$

∃ 2 sign changes for 1st column

⇒ 2 roots on right half of s-plane.

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eg (3) :- for $s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1$

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$

s^5	1	3	2
s^4	2	6	1
s^3	0	1.5	0
s^2	0		
s			
1			

→ ∴ 2 zero in 1st column ⇒ Routh criterion fails here

AVOID failure

eg (2). Continued -

$s^3 + 4s^2 + s + 16$

$a_0 = 1, \; a_1 = 4, \; a_2 = 1, \; a_3 = 16$

s^3	1	1	0	0
s^2	4	16	0	0
s	b_1	b_2	0	0
s^0	c_1	c_2	0	0

→ the later terms are zero (taken) always.

$$b_1 = \frac{\begin{vmatrix} 1 & 1 \\ 4 & 16 \end{vmatrix} \times (-1)}{4} = -3$$

$$b_2 = \frac{\begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix} \times (-1)}{4} = 0$$

$$c_2 = \frac{\begin{vmatrix} 4 & 0 \\ -3 & 0 \end{vmatrix} \times (-1)}{-3} = 0$$

$$c_1 = \frac{\begin{vmatrix} 4 & 16 \\ -3 & 0 \end{vmatrix} \times (-1)}{-3} = 16$$

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So, matrix becomes

1	1	0
4	16	0
-3	0	0
16	0	0

The no. of sign changes are :-

from ⊕4 to ⊖3 → 1st

⊖3 to ⊕16 → 2nd

⇓

2 sign changes

So, there will be 2 roots on right half of s-plane.

★ We saw Routh Criteria/array to fail when ∃ a zero in the first column.

If the 1st Element of any ~~row~~ Routh array becomes zero :-

Method 1

S1) Substitute a small +ve no., ϵ for the zero & proceed to evaluate the rest of the Routh array.

S2) Examine the signs of 1st column by letting $\epsilon \to 0$.

S1)

s^5	1	3	2
s^4	2	6	1
s^3	ϵ	1.5	0
s^2	$\frac{6\epsilon-3}{\epsilon}$	1	
s	$\frac{1.5(6\epsilon-3)-\epsilon}{\frac{6\epsilon-3}{\epsilon}}$	0	
s^0	1		

S2) $\lim_{\epsilon \to 0} \frac{6\epsilon-3}{\epsilon} = 6 - \lim_{\epsilon \to 0} \frac{3}{\epsilon}$

$= 6 - \infty = -\infty$

$\lim_{\epsilon \to 0} \frac{1.5(6\epsilon-3)-\epsilon^2}{6\epsilon-3}$

$= \frac{-4.5}{-3} = 1.5.$

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So, re-writing the array:

	1	3	2	
	2	6	1	
+ve (assumed) ←	⊕ ε	1.5	0	
	−∞	1	0	
	+1.5	0		
	1	0		

∃ 2 sign changes

⊕ ε ⇝ ⊖ ∞

& ⊖ ∞ ⇝ ⊕ 1.5

So, 2 roots on right half of s-plane

Method 2 If ∃ zero's in the 1st column of Routh array;

S1) Replace $s \rightarrow 1/z$ in characteristic eqⁿ

S2) Taking LCM, rearrange characteristic eqⁿ in descending powers of z

In eq (3)

We had : $s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

S1) $s \rightarrow 1/z$

$$\Rightarrow \frac{1}{z^5} + \frac{2}{z^4} + \frac{3}{z^3} + \frac{6}{z^2} + \frac{2}{z} + 1 = 0$$

$$\Rightarrow 1 + 2z + 3z^2 + 6z^3 + 2z^4 + z^5 = 0$$

S2) $\Rightarrow z^5 + 2z^4 + 6z^3 + 3z^2 + 2z + 1 = 0$

↳ new characteristic eqⁿ

$a_0 = 1$ $\quad a_5 = 1$

$a_1 = 2$

$a_2 = 6$

$a_3 = 3$

$a_4 = 2$

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Routh array

z^5	1	6	2	0
z^4	2	3	1	0
z^3	4.5	1.5	0	0
z^2	2.33	1	0	0
z	-0.431	0	0	0
1	+1	0	0	0

$$c_1 = \frac{\begin{vmatrix} 2 & 3 \\ 4.5 & 1.5 \end{vmatrix} \times (-1)}{4.5}$$

$$= \frac{13.5 - 3}{4.5}$$

$$= \frac{10.5}{4.5} \quad \frac{7}{3} \quad 2.33$$

$$d_1 = \frac{\begin{vmatrix} 4.5 & 1.5 \\ 2.33 & 1 \end{vmatrix} \times (-1)}{2.33}$$

$$z = -0.431$$

Seeing the 1st column,
$\exists$ 2 sign changes.

(+) 2.33 → (−) 0.431
(−) 0.431 → (+) 1

So, $\exists$ 2 roots on right half of s-plane.
(same result as in Method 1)

* SPECIAL CASE:

If all elements of a row are zero :-
$\exists$ non availability of coeff. in that row.

eg:

s^5	a	b	c
s^4	0	0	0
⋮			

eg (2) :- Characteristic eq^n

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

$a_0 = 1$ $\quad a_3 = 12$ $\quad a_6 = 16$

$a_1 = 2$ $\quad a_4 = 20$

$a_2 = 8$ $\quad a_5 = 16$

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s^6	1	8	20	16
s^5	2	12	16	
s^4				
s^3				
s^2				
s				
1				

$R_2 \longrightarrow R_2 \div 2$ (we can divide any row by a constt, if reqd)

s^6	1	8	20	16
s^5	1	6	8	
s^4	~~2~~ 1	~~12~~ 6	~~16~~ 8	$R_3 \div 2$
s^3	0	0	0	
s^2				
s				
1				

∃ zero's in R4.

So, take the row R_3 & write its eqⁿ

$A(s) = s^4 + 6s^2 + 8 = 0$: AUXILIARY EQⁿ.

Now $\frac{dA}{ds} = 4s^3 + 12s = 0$

or $s^3 + 3s = 0 \quad (\div 4)$

Now, this eqⁿ that we get ($s^3 + 3s = 0$), its coeff. should be used to fill up R4, instead of the zero's present.

So, matrix becomes

PROCEDURE

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S^6	1	8	20	16
S^5	1	6	8	
S^4	1	6	8	
S^3	1	3	0	
S^2	3	8	0	
S	1/3	0		
1	8	0		

Clearly, from Col. I,
$\exists$ no sign change
So, its a stable sys.

Now, we need to find roots of the Auxiliary eq^n.
$\because$ roots of auxiliary eq^n are also roots of original characteristic eq^n.
These roots are the most dominant roots of original charac. eq^n from stability pt. of view.

Hence, $A(s) = s^4 + 6s^2 + 8 = 0$

Let $x = s^2$, $x^2 = s^4$

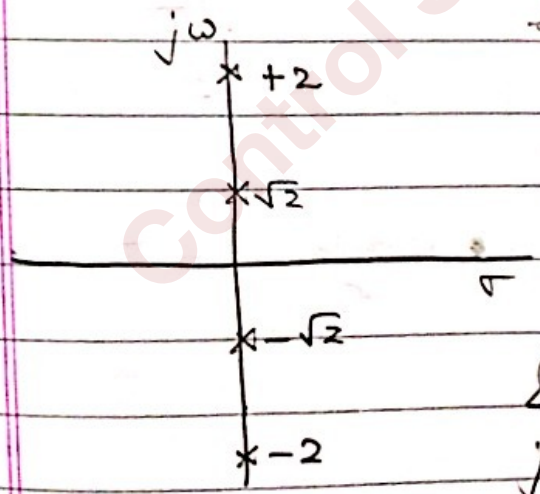
$\Rightarrow x^2 + 6s + 8 = 0$

$\Rightarrow x^2 + 4s + 2s + 8 = 0$

$\Rightarrow x(x+4) + 2(x+4) = 0$

$\Rightarrow s^2 = -2, -4$

$\Rightarrow s = \pm\sqrt{2}j, \pm 2j$



Since $\exists$ non repeated roots on jω axis $\Rightarrow$ sys is limitedly stable or marginally stable.

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Q. $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0$

$a_0 = 1$ $\quad a_3 = 2$

$a_1 = 1$ $\quad a_4 = 3$

$a_2 = 2$ $\quad a_5 = 15$

s^5	1	2	3	0
s^4	1	2	15	0
s^3	0	-12	0	
s^2	∞	∞		
s				
1				

→ 1st element became zero

So, use Method (2) / Method 1

Method 1) $0 \rightsquigarrow \epsilon$

S1)

s^5	1	2	3			1
s^4	1	2	15			1
s^3	ϵ	-12	0			ϵ
s^2	$\frac{2\epsilon+12}{\epsilon}$	15				$+\infty$
s	$\frac{-12(2\epsilon+12)-15\epsilon}{2\epsilon+12/\epsilon}$	0	0			-12
1	15	0	0			15

S2) $\lim_{\epsilon \to 0} \frac{2\epsilon+12}{\epsilon} = 2 \lim_{\epsilon \to 0} \frac{\epsilon+6}{\epsilon} = 2 + \lim_{\epsilon \to 0} \frac{6}{\epsilon}$

$= 2 + \infty$

$\rightarrow +\infty$ (+ve)

$$\lim_{\epsilon \to 0} \frac{-12\left(\frac{2\epsilon+12}{\epsilon}\right) - 15\epsilon}{\frac{2\epsilon+12}{\epsilon}} = \lim_{\epsilon \to 0} \left[\frac{-24\epsilon - 144 - 15\epsilon^2}{2\epsilon + 12}\right]$$

$= -12$ (-ve)

So, $\exists$ 2 sign changes $\Rightarrow$ 2 roots on RHS (unstable)

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Method 2 :-

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0$$

$s \to 1/z$

$$\Rightarrow \frac{1}{z^5} + \frac{1}{z^4} + \frac{2}{z^3} + \frac{2}{z^2} + \frac{3}{z} + 15 = 0$$

$$\Rightarrow 15z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0$$

↳ new characteristic eqⁿ.

$a_0 = 15$ $\quad a_3 = 2$

$a_1 = 3$ $\quad a_4 = 1$

$a_2 = 2$ $\quad a_5 = 1$

z^5	15	2	1
z^4	3	2	1
z^3	-8	-4	0
z^2	1/2	+1	0
z	12	0	0
1	+1	0	0

∃ 2 sign changes

⇒ 2 roots on RHS ⇒ unstable.

= same as in Method 1

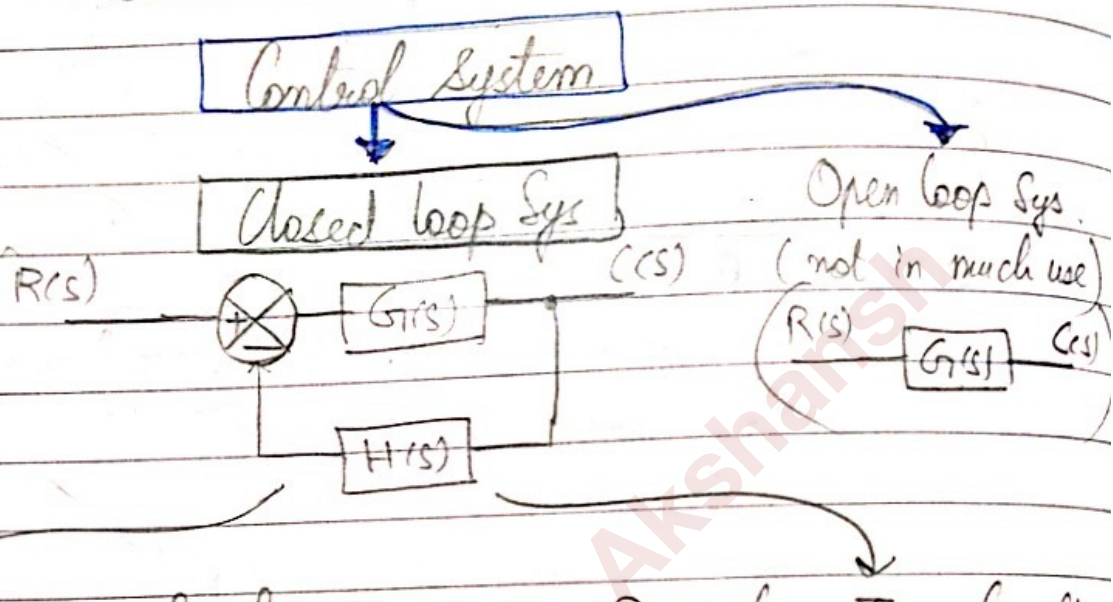
Q. OLTF of a unity feedback sys. is

$$G(s) = \frac{k}{(s+2)(s+4)(s^2+6s+25)}$$

Applying Routh criterion, discuss stability of given CL sys. as a fn of k.

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✡ AVOIDING CONFUSION



Closed loop Transfer fn

→ $TF = \frac{K\,G(s)}{1 + G(s)H(s)}$

→ Characteristic eqⁿ:
$1 + G(s)H(s) = 0$.

→ Considering the TF:
Numerator, $kG(s) = 0$
↳ values got: called as Zero's
Denominator, $1 + G(s)H(s) = 0$
↳ values got: called as Poles

→ Zero's + Poles: Collectively called: Roots.

Open loop Transfer fn

→ $TF = G(s)H(s)$
↳ for unity f/b:
$TF = G(s)(1) = G(s)$

→ Characteristic eqⁿ:-
$1 + G(s)H(s) = 0$.

✡ Symbol for Poles : ×
Symbol for Zero's : ○

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K: open loop gain.

Solution :- $1 + G(s) = 0$.

$\Rightarrow s^4 + 12s^3 + 69s^2 + 198s + (200 + K) = 0$.

s^4	1	69	$200 + K$
s^3	12	198	0
s^2	52.5	$200 + K$	
s	$\frac{10395 - 12(200+K)}{52.5}$		
1	$200 + K$		

Now, for sys to be stable, the first elemt. of Routh Array has to be of the same sign (in this case +ve)

So, $\frac{10395 - 12(200+K)}{52.5} > 0$ & $200 + K > 0$

$\Rightarrow K < 666.25$ $\Rightarrow [K > -200]$

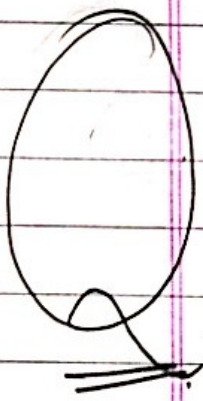
So, $\boxed{-200 < K < 666.25}$

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ROOT- LOCUS TECHNIQUE

- a graphical technique; when k is varied from 0 to ∞, what happens to the roots (poles) of characteristic eqⁿ.
- k: open loop gain
- technique given by E.R Evans.
- Root locus: location of roots of * charac. eqⁿ as k varies from 0 → ∞.



Draw (RL) (Root Locus) diagram for a CL sys. whose OLTF, $G(s)H(s) = \frac{k}{s(s+5)(s+10)}$

Also, comment on its stability.

[Reminder :- Characteristic eqⁿ of OLTF = $1 + G(s)H(s) = 0$]

✱ RULES FOR CONSTRUCTION OF ROOT LOCUS:

Rule 1: The RL is always symmetrical about real axis (x-axis). Roots can be: (i) real (ii) complex (iii) both.

Rule 2: $G(s)H(s)$ = OLTF of the sys,

P = no. of OL poles
Z = no. of OL zero's
n : no. of branches of RL

} denotions

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(Check that your problem belongs to which case)

Case 1	Case 2
$P > Z$; $n = P$	$Z > P$; $n = Z$
• Branches start from each loc^n of OL pole.	• Branches terminate at each of finite loc^n of OL ~~pole~~ zero.
• Z no. of branches will terminate at loc^{ns} of OL zero's	• P no. of branches will start from OL poles.
• P-Z will approach infinity.	• Z-P will originate from ∞ & terminate at finite zero's.

The branch dir^n is always from
OL Poles $\longrightarrow$ OL Zero's.

In the given problem, poles are got as,
denominator = 0.
Zeros are got as
numerator = 0.

So, $s(s+5)(s+10) \Rightarrow s = 0, -5, -10$
(3 poles) $\Rightarrow P = 3$

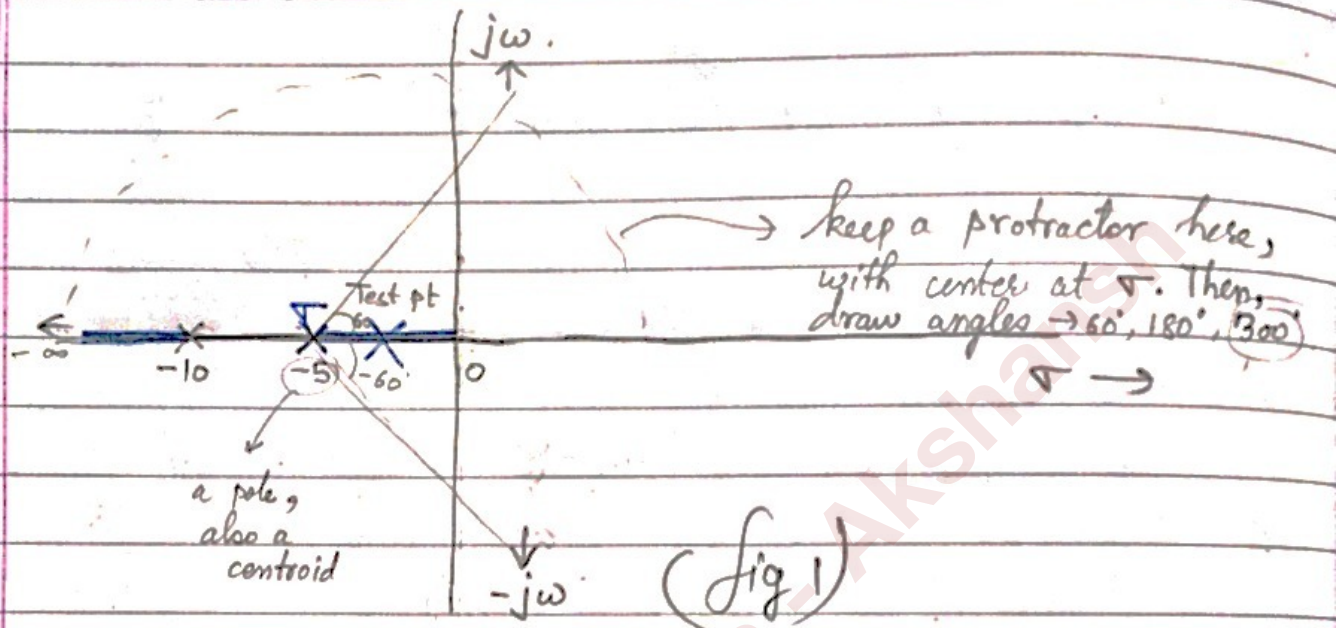
no. of zeros = 0. $\Rightarrow Z = 0$

So, $P > Z \Rightarrow$ Case (1).
$\Downarrow$
$n = P$.

So, no. of branches = $n = 3$.

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By the properties of case (1), all the 3 branches $\rightarrow \infty$.



(fig 1)

Rule 3: Root locus branch on real axis:

A pt. on the real axis lies on RL if the sum of OL Poles & OL Zeros on the real axis, to the right of that point is ODD.

for our problem, take any pt. on real axis (Test pt.) as shown. Seeing the right side of it, we find only 1 pt ($s=0$) ($1 \equiv$ odd). So, from 0 to -5 is one branch as shown. Hy, from -10 to $-\infty$ is another branch. Out of these 2 branches, -10 to $-\infty$ is the infinite branch.

- complex conjugates shouldn't be considered while using this rule.

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Rule 4: Generally, in most of the systems, $P > Z$, hence, "$P-Z$" are the no. of branches that'll approach ∞.

$P-Z$ branches approach ∞ along st. lines called asymptotes.

Angle of asymptote,

$$\phi = \frac{(2q+1)\,180^\circ}{P-Z}$$

↳ q: # no. of branches that approach ∞ = n. then $q = 0, 1, \ldots, n-1$

for $P-Z = 3$, $q = 0, 1, 2$

for our problem,

for $q = 0$, $\phi = 60^\circ$

$q = 1$ $\phi = 180^\circ$

$q = 2$ $\phi = 300^\circ$

Rule 5: Centroid (σ):

All asymptotes intersect the real axis at a common pt known as Centroid (σ).

$$\sigma = \frac{\sum \text{R.P of poles} - \sum \text{R.P of zero's}}{P-Z}$$

(R.P → Real parts)

$$= \frac{[0+(-5)+(-10)] - 0}{3-0} = \frac{-15}{3} = -5.$$

With centroid known, draw the angles of asymptotes on the graph, as shown.

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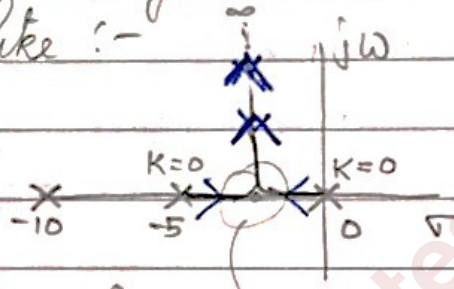
Rule 6. Break away pt. / break in point

- **Break away pt.** : (for a pole)
  A pt. on the root locus where multiple roots of characteristic eq^n occurs for a particular value of k.

adjacent roots

$\therefore \exists$ 3 branches $(P-Z=3)$ & in the graph, we have only 2 branches, so, actually, they are like :-

i.e., $\exists$ 2 adjacent poles on real axis :-



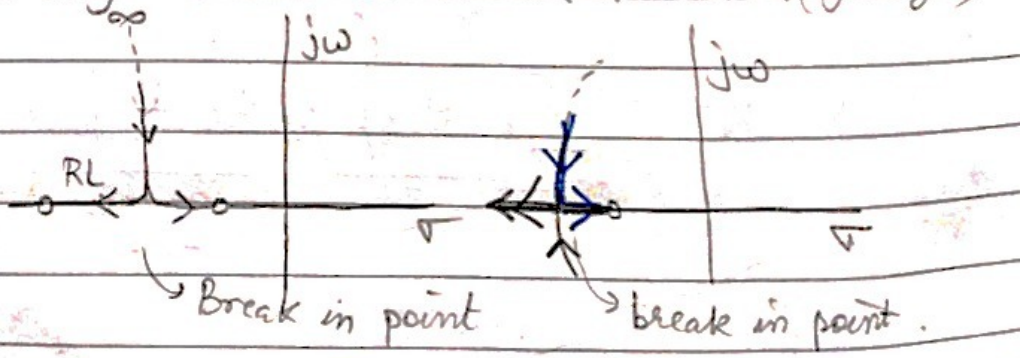
(fig 2)

then, $\exists$ break away pt (maybe)

Break away pt

K increases from both sides. At a particular value of K, the roots merge & go towards ∞. The pt. of merging is called break away pt.

Why, point going from infinity to a pt & then separating ⟶ BREAK IN POINT. (for a zero)



($\exists$ a zero on σ axis & entire real axis is the RL line).

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✱ Steps to determine coordinates of break away pts.

S1) Construct characteristic eqⁿ, $1 + G(s)H(s) = 0$

S2) From the eqⁿ, separate the terms involving k & terms involving s.

$$k = f(s).$$

S3) Differentiate the above eqⁿ w.r.t 's' & equate it to zero.

$$\frac{dk}{ds} = 0.$$

S4) Roots of above eqⁿ gives break away points.

for our problem,

$$G(s)H(s) = \frac{k}{s(s+5)(s+10)}$$

⇒ S1) $1 + G(s)H(s) = 0$

$$\Rightarrow 1 + \frac{k}{s(s+5)(s+10)} = 0$$

$$\Rightarrow \frac{(s^2+5s)(s+10) + k}{s(s+5)(s+10)} = 0$$

$$\Rightarrow s^3 + 15s^2 + 50s + k = 0.$$

S2) $$\Rightarrow k = -s^3 - 15s^2 - 50s.$$

S3) $$\frac{dk}{ds} = 0$$

$$\Rightarrow -3s^2 - 30s - 50 = 0 \Rightarrow 3s^2 + 30s + 50 = 0$$

S4) $$\Rightarrow s = \frac{-30 \pm \sqrt{900 - 600}}{6} \simeq -2.113, -7.88$$

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Out of these values, the value lying b/w (0 & -5) or (-5 & -10) should be ans.

So, -7.88 : rejected.

So, the break away pt is -2.113

* If both pts are coming on the Root locus, So, check for the value of k at that pt. If k is +ve, take that root (for proving $k > 0$, substitute the value of k in the characteristic eq^n)

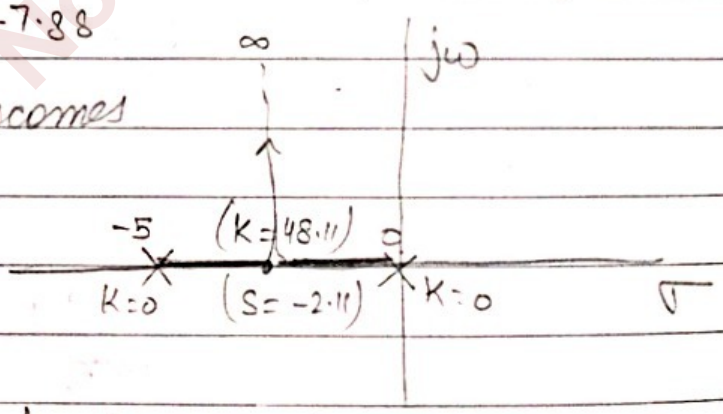
Checking for $s = -2.113$.

$$k = -(-2.113)^3 - 15(-2.113)^2 - 50(-2.113)$$
$$= 48.11 \ (+ve)$$

$$k\big|_{s=-7.88} = -48.11 \ (-ve)$$

So, the fig. becomes

(fig 3)



* If both $k < 0$: no breakaway pt

$k > 0$: 2 breakaway pts.

Rule 7: Applicable only if a complex pole (or complex zero) is present in your sys.

For our problem, this rule isn't applicable.

Angle of departure (for a complex pole)

$$\phi_d = 180° - (\phi_p - \phi_z)$$

Angle of arrival (for a complex zero)

$$\phi_A = 180° - (\phi_z - \phi_p)$$

Control Systems Notes - Aksmansh

Rule 8: The intersection of root locus branches with imaginary axis

↳ This has to be found:

Already done in Rule 6 → S1

S1) Construct charac. eqⁿ, $1 + G(s)H(s) = 0$.

S2) Apply Routh criterion

S3) Determine the marginal value of k.

S4) Construct auxiliary eqⁿ $A(s) = 0$

S5) Roots of this auxiliary eqⁿ are the intersection points of RL with $j\omega$ axis (imaginary axis)

For our problem, charac. eqⁿ :-

S1) $s^3 + 15s^2 + 50s + k = 0$.

S2)

s^3	1	50
s^2	15	k
s	$(750-k)/15$	0
1	k	

S3) $\frac{750-k}{15} > 0$; $k > 0$

$\Rightarrow k = 750$.

S4) $A(s) = 15s^2 + k = 0$ (written using R2, by keeping $R_3 = 0$)

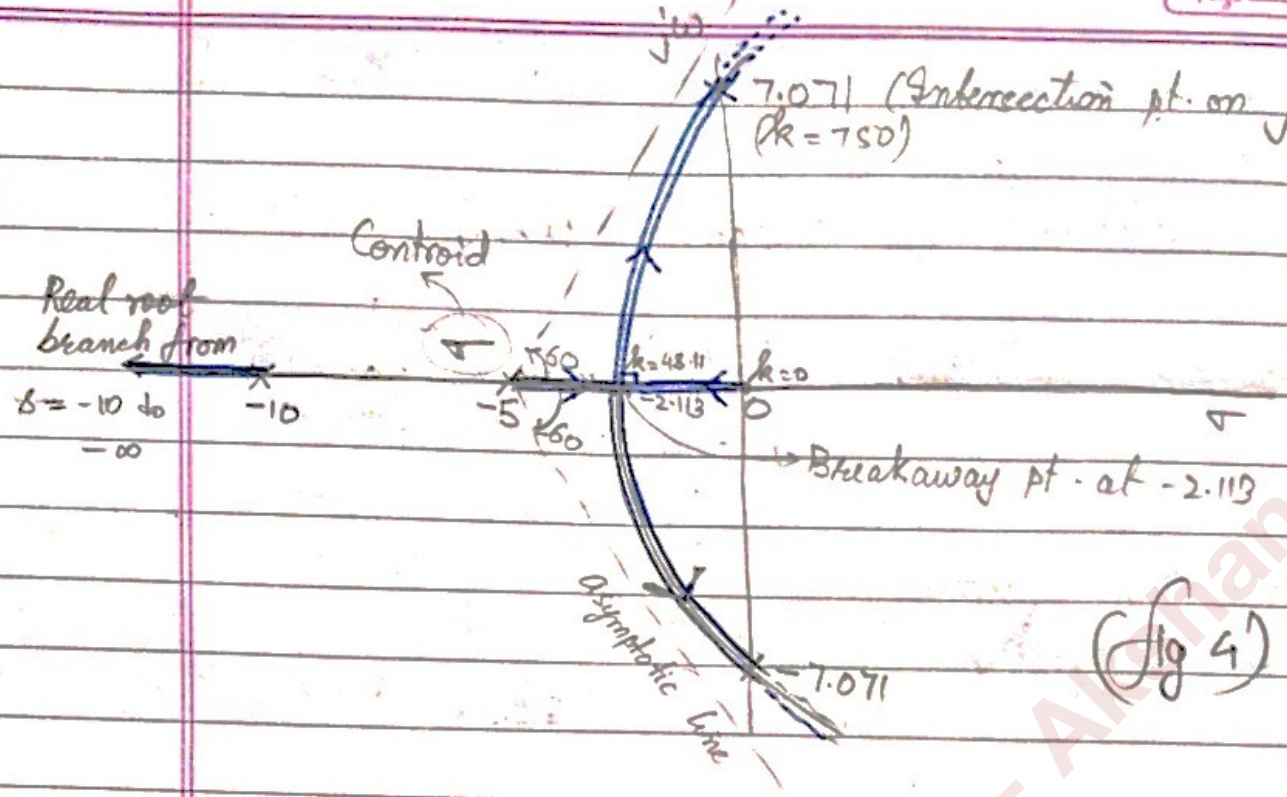
$\Rightarrow 15s^2 + 750 = 0$

$\Rightarrow s = \pm j(7.071)$

From fig 14) (next pg), by the rules, there should be 3 branches & all of them should extend to infinity. So,

} The 3 infinite branches, as in fig 4)

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(fig 4)

★ Conclusion/Summary (to be written ∀ problems)

(a) for $0 < k < 750$, sys is stable, as RL is in left half of s plane

(b) $k = 750$, marginally stable.

(c) $750 < k < \infty$, sys is unstable, dominant roots lie on right half of s-plane.

Dominant roots: (roots closest to jω axis; $s = 0, -5$)

Find RL for given OLTF

$$G(s)H(s) = \frac{k(s+5)}{s^2+4s+20}$$

Sl) Finding zero's :- $s+5 = 0 \Rightarrow s = -5 \Rightarrow z = 1$

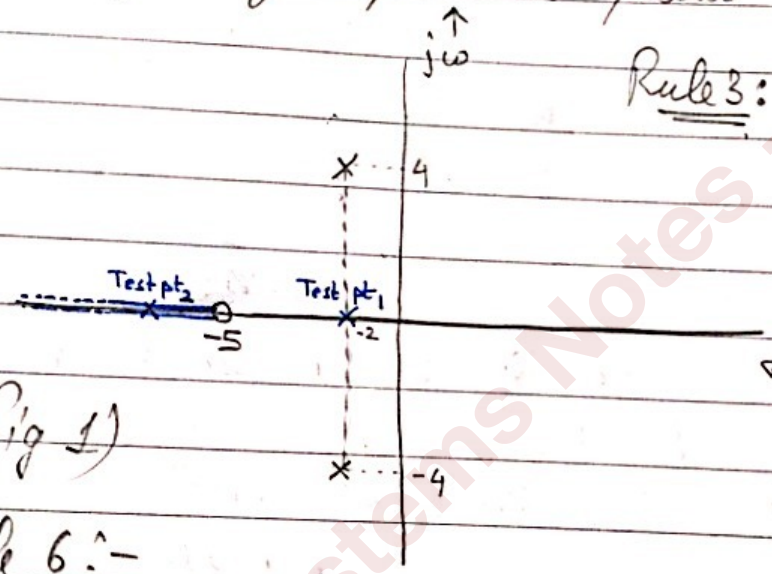
poles :- $s^2+4s+20 = 0$

$\Rightarrow s = -2 \pm 4j \Rightarrow P = 2$

Control Systems Notes - Alphansh

By rule 2,
$P > Z$.
$\Rightarrow n = 2$ = no. of branches of RL
And,
no. of branches terminating at ∞ =
$P - Z = 2 - 1 = 1$

S2) Draw pole zero plot & predict breakaway pt



(Fig 1)

Rule 3: Taking a test pt at $s = -2$, say. On right of it, no. of poles/zero's are zero
So, even.
Taking another test pt at $s = -7$, say. On right of it, no. of poles/zeros = 1
odd.
So, its a part of RL branch.

Rule 6:-
If ∃ a zero on x axis, & beyond that pt, ∃ an infinite RL branch, then, there would be a break in pt. on the line.

S3) Finding asymptotes
no. of asymptotes $= P - Z = 1$
Angle of asymptotes $= \phi = \frac{[2(0)+1]180}{1} = 180°$

Control Systems Notes - Akshansh

S4) Find Centroid

$$\sigma = \frac{\sum(\text{R.P of poles}) - \sum(\text{RP of zeros})}{P-Z}$$

$$= \frac{[(-2)+(-2)] - (-5)}{2-1}$$

$$\Rightarrow \sigma = \frac{-4+5}{1} = \boxed{1}$$

S5) Find break in pt by solving $1 + G_{(s)}H_{(s)} = 0$.

$$1 + G_{(s)}H_{(s)} = 1 + \frac{k(s+5)}{s^2+4s+20} = 0$$

$$\Rightarrow s^2+4s+20 + k(s+5) = 0$$

$$\Rightarrow k = \frac{-s^2-4s-20}{s+5}$$

$$\Rightarrow \frac{dk}{ds} = \frac{(s+5)(-2s-4) - (-s^2-4s-20)(1)}{(s+5)^2} = 0$$

$$\Rightarrow s^2+10s = 0 \Rightarrow s(s+10) = 0$$

$$\Rightarrow s = 0, -10$$

Checking for k = +ve or -ve.

When $s = 0$, $K = -4$

$s = -10$, $k = 16$ (+ve).

So, $s = -10$ is a break-away pt.

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Note:- $k > 0$: intersection with jω axis ∃

$k < 0$: intersection with jω axis ∄

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S6) Finding intersection pt of RL with jω axis.

M1) Charac eq^n :- $s^2 + 4s + 20 + ks + 4k = 0$

$\Rightarrow s^2 + (4+k)s + 20 + 5k = 0$

s^2	1	$20+5k$
s	$4+k$	0
1	$20+5k$	

First term with k, > 0

$\Rightarrow 4 + k > 0 \Rightarrow k = -4$

$= -ve$

$\Rightarrow$ no intersection with jω axis.

M2) : To find intersection pts :-

If Routh Array not used

Substitute $s \rightarrow j\omega$ in characteristic eq^n.

So, we get

$(j\omega)^2 + (4+k)(j\omega) + (20+5k) = 0$

$\Rightarrow -\omega^2 + 4j\omega + jk\omega + 20 + 5k = 0$

$\Rightarrow (-\omega^2 + 20 + 5k) + j(4\omega + k\omega) = 0$

equating real & imaginary parts on both sides, we get

$-\omega^2 + 20 + 5k = 0$ & $4\omega + k\omega = 0$

$\Rightarrow (-0)^2 + 20 + 5K = 0$ (∵ $\omega = 0$) $\Rightarrow \omega = 0$ or $k = -4$

$\Rightarrow K = -4$ (as above) $\Rightarrow$ No intersection with jω axis.

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S7) From step 1, we saw $s = -2 \pm j4$

There are complex poles.

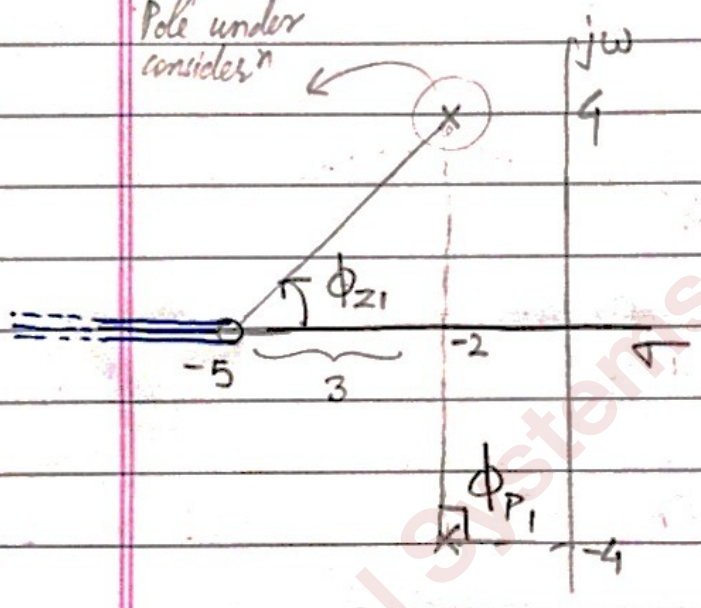
For complex poles, we find angle of departure $\angle\theta_d$ by Rule 7,

ϕ_P : angles from all other poles to the zero/pole we are considering -

ϕ_Z : angle from all zeros to the pole/zero under consideration -

Note :- ↻ dir^n :- +ve

↺ dir^n :- −ve



$\phi_{P_1} = 90^\circ$

$\phi_{Z1} = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$.

$\phi_d = 180^\circ - (\phi_P - \phi_Z)$

$= 180 - (90 - 53.13^\circ)$

$\phi_d\Big|_{\text{at } -2+4j} = 143.13^\circ$.

* Note: If ∃ more than one pole/zero, then formula changes as:

$\phi_d = 180^\circ - (\phi_{P_1} + \phi_{P_2} + \dots$

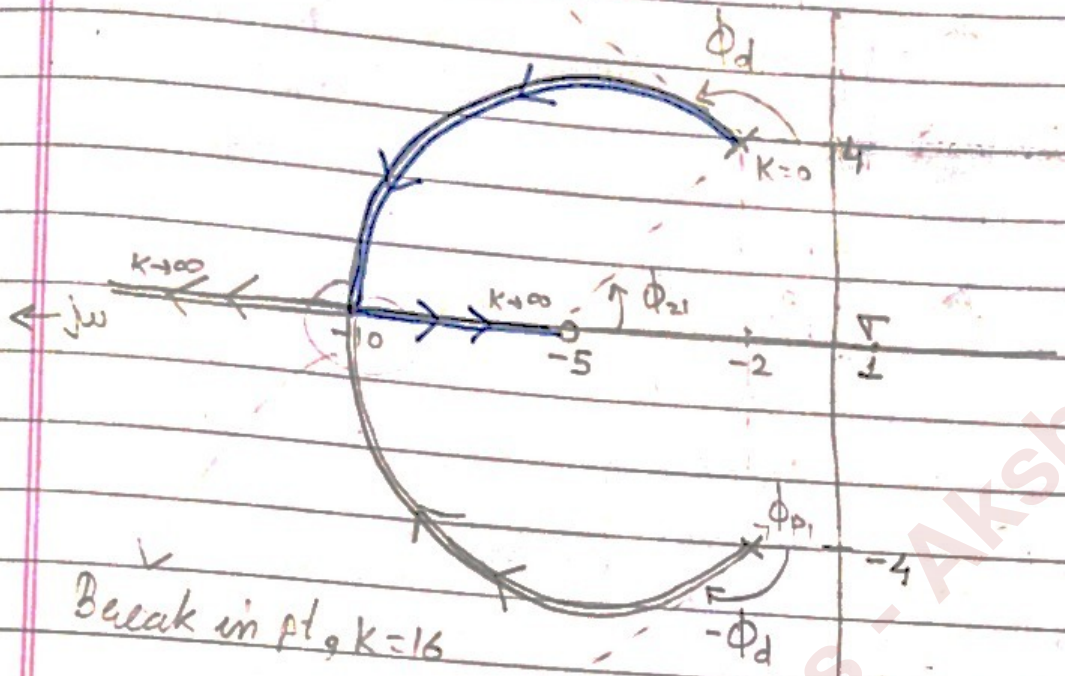
$* - (\phi_{Z_1} + \phi_{Z_2} + \dots))$

Ily, $\phi_d\Big|_{\text{at } -2-4j} = -\text{ve of } \phi_d\Big|_{-2+4j}$ (by symmetry)

$\Rightarrow \phi_d\Big|_{-2-4j} = -143.13^\circ$.

Control Systems Notes - Akshansh

S8) Making the final plot



Break in pt, k=16

Q.

Given :- $G_1(s) = \frac{k}{s(s^2+4s+13)}$

S1) $z = 0$ (no zeros)

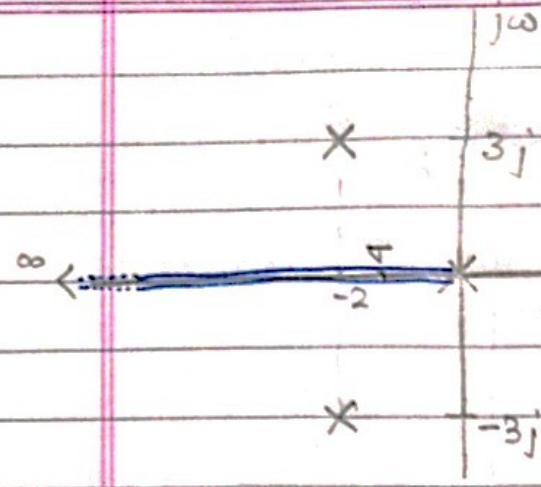
$P = 3$ $\left(s = 0,\ s = \frac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3j\right)$

$n = P = 3$ = no. of branches of root locus

No. of branches of root locus terminating at $\infty = P - Z = 3$.

S2) To find root locus & prediction of break away pt.

Control Systems Notes - Akshansh



Take a test pt. at $\sigma = -1$, Say, on right, ∃ 1 pole (odd) ⇒ part of RL

S3) Finding asymptotes

$$\phi = \frac{180(2q+1)}{P-Z}$$

q : no. of branches terminating at ∞ (= 3)

↳ for $q = 0, 1, 2$ i.e., $q = 0, 1, 2$

$\phi = 60°, 180°, 300°$

S4) Centroid, $\sigma = \frac{\sum \text{R.P of poles} - \sum \text{R.P of zeros}}{P-Z}$

$$= \frac{[0 + (-2) + (-2)] - (0)}{3}$$

$$\Rightarrow \sigma = \frac{-4}{3} = -1.33$$

S5) Finding Break away pt.

There are 3 cond^ns for existence of break away/break in pt (Rule 6: See the 3 figures).

Here, none of the cond^ns are there

⇒ no break away point.

Control Systems Notes - Akshansh

S6 Intersection of RL with jω axis

Finding characteristic eq^n:-

$1 + G(s)H(s) = 0$

$\Rightarrow 1 + \frac{k}{s(s^2+4s+13)} = 0$

$\Rightarrow s^3 + 4s^2 + 13s + k = 0$

$a_0 \quad a_1 \quad a_2 \quad a_3$

s^3	1	13	0
s^2	4	k	0
s	$\frac{52-k}{4}$	0	0
1	k	0	0

first term with $k > 0$ (for marginal value)

$\Rightarrow 52 - k = 0$

$\Rightarrow k = 52$

Auxiliary eq^n:

$A(s) = 4s^2 + k = 0$

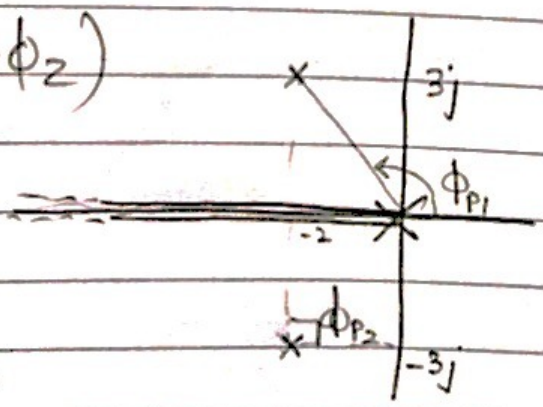
$= 4s^2 + 52 = 0$

$\Rightarrow s = \pm j\sqrt{13}$

S7) Finding angle of departure:

$\phi_d = 180 - (\phi_p - \phi_z)$

for $-2+3j$



$\phi_p = \phi_{P_1} + \phi_{P_2}$

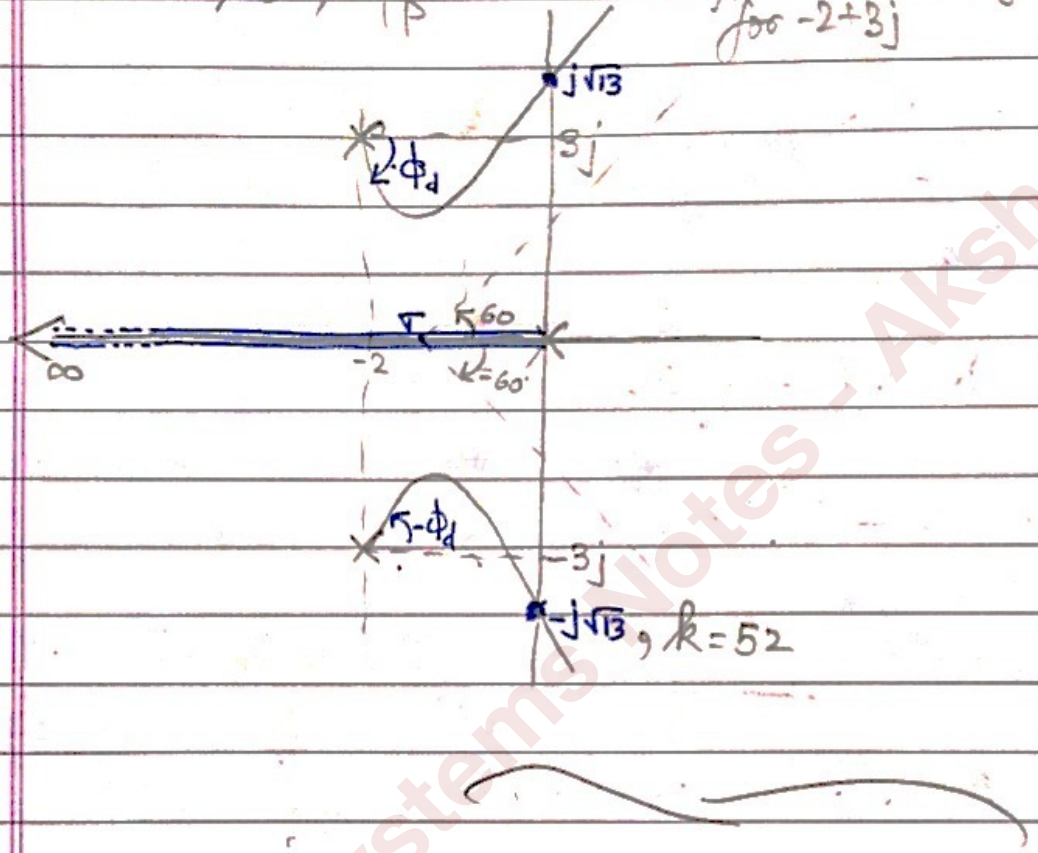
$\phi_z = 0$

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Now, $\phi_{P_1} = 90 + \tan^{-1}\left(\frac{2}{3}\right)$

$\phi_{P_2} = 90^\circ$

So, $\phi_P = -33.67^\circ$ for $-2+3j$ $\quad$ Mag. $\phi_d\Big|_{-2-3j} = 33.69$



Q.

$$G(s)H(s) = \frac{k}{s(s+1)(s+2)(s+3)}$$

Sol) Zero's, $Z = 0$

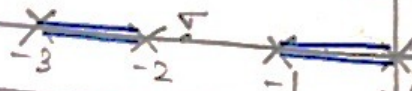
Poles, $P = 4$ $\quad (s = 0, -1, -2, -3)$

$n = P = 4$

No. of branches extending at $\infty = P - Z = 4$

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S2)



: Tells about the RL branch (part of it)

Seeing from Rule 6, ∃ 2 break away pts. (b/w -3 & -2 & b/w -1 & 0)

S3) Asymptotes :-

$$\phi = \frac{180(2q+1)}{4}$$

↳ $q = 0 \Rightarrow \phi = 45^\circ \quad (\phi_1)$

$q = 1 \Rightarrow \phi = 135^\circ \quad (\phi_2)$

$q = 2 \Rightarrow \phi = 225^\circ \quad (\phi_3)$

$q = 3 \Rightarrow \phi = 315^\circ \quad (\phi_4)$

S4) Centroid,

$$\sigma = \frac{[0 + (-1) + (-2) + (-3)] - 0}{4}$$

$$\Rightarrow \sigma = \frac{-7}{4} = -1.85$$

Control Systems Notes - Akshansh

5) Finding break away pts :-

$$1 + G(s)H(s) = 0$$

$$\Rightarrow s(s+1)(s+2)(s+3) + k = 0$$

$$\Rightarrow k = -s(s+1)(s+2)(s+3)$$

$$\frac{dk}{ds} = -\left[(s+1)(s+2)(s+3) + s(s+2)(s+3) + s(s+1)(s+3) + s(s+1)(s+2)\right]$$

$$\text{or } k = -(s^3+2s^2+s^2+2s)(s+3)$$

$$= -(s^4+2s^3+s^3+2s^2+3s^3+6s^2+3s^2+6s)$$

$$\Rightarrow k = -s^4-6s^3-11s^2-6s$$

$$\Rightarrow \frac{dk}{ds} = -4s^3-18s^2-22s-6 = 0$$

$$\Rightarrow \frac{dk}{ds} = 2s^3+9s^2+11s+3 = 0.$$

By trial & error :-

one of the root :- -1.5

-1.5	2	9	11	3
	0	-3	-9	-3
	2	6	2	0

$$\Rightarrow 2s^2+6s+2 = 0$$

$$\Rightarrow s^2+3s+1 = 0$$

$$\Rightarrow s = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

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So, roots are :- $-1.5, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$

:- $-1.5, -0.38, -2.61$

Using these values to find value of K

s	k
-1.5	-ve
-0.38	1 (+ve)
-2.61	1 (+ve)

→ 2 values of s gives k +ve.
So, ∃ 2 break-away pts.

Idea to guess break away pt :- See fig. ∃ symm about Root Locus

S6 Intersection with jω axis

whole row ← ÷ by 6

s^4	1	11	k
s^3	1	1	0
s^2	10	k	0
s	$\frac{10-k}{10}$	0	0
1	k		

Characteristic eq^n :-
$s^4 + 6s^3 + 11s^2 + 6s + K = 0$

Seeing 1st row with
$k > 0$

$\Rightarrow \frac{10-k}{10} > 0$

$\Rightarrow 10 > k$

$\Rightarrow k < 10$

& $k > 0$.

for marginal cond^n ← ↓

$k = 10$

Auxiliary eq^n → $10s^2 + k = 0$

$\Rightarrow 10s^2 + 10 = 0$

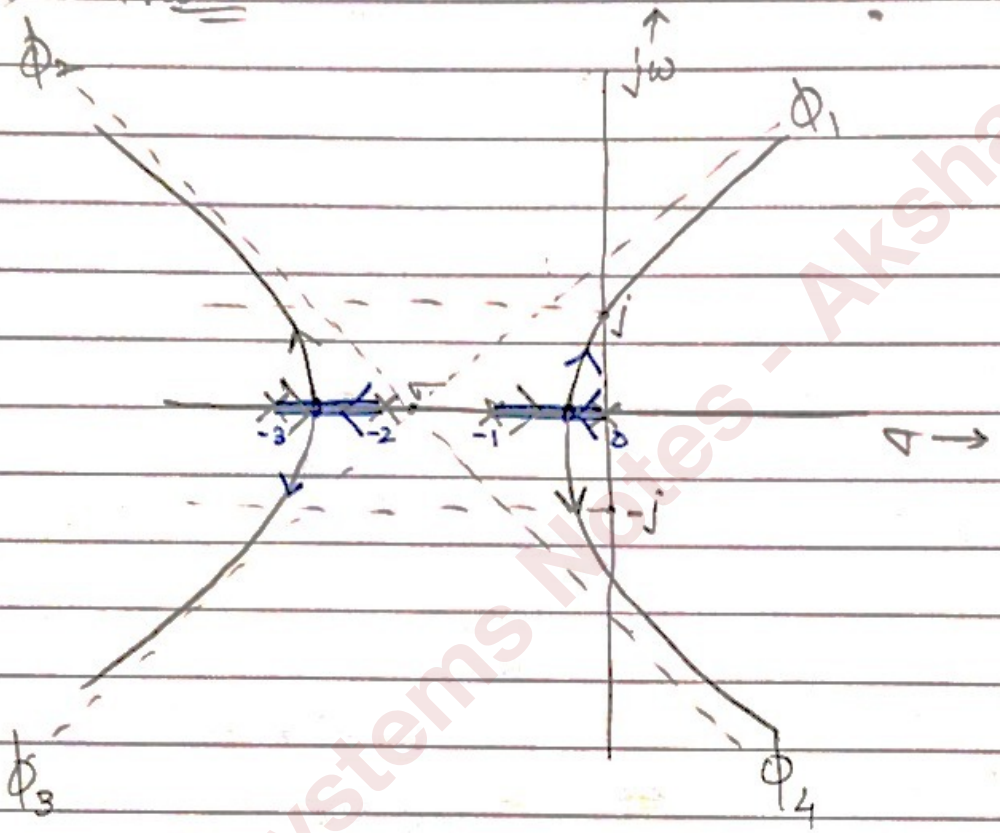
$\Rightarrow s = \pm j$ → Intersection on jω axis

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S7) Angle of arrival / departure :-

(Not applicable)

S8) Plot



Idea :- • Mark centroid & using protractor, mark line of asymptotes

• See for the places having RL

• Show break away pt in graph.

$s = -0.38$ & -2.61 → break away pts.

• Join break away pt & pt. intersecting jω axis & plot.

• Do rest by symmetry.

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Puffin
Date 6/5/13
Day

Bode Plot

- Frequency response plot of sinusoidal TF of a sys.
- It consists of 2 graphs
  - magnitude plot
  - phase angle plot
- multiples of magnitude can be converted to addⁿ

- $G(s) = \frac{k(1+sT_1)}{s(1+sT_2)(1+sT_3)}$ → unit term; → numerator with order 1; → denominators with order 1 each

$$= \frac{k(1+j\omega T_1)}{j\omega(1+j\omega T_2)(1+j\omega T_3)}$$

$$= \frac{(k\angle 0^\circ)\left(\sqrt{1+\omega^2T_1^2}\ \tan^{-1}(\omega T_1)\right)}{(\omega\angle 90^\circ)\left(\sqrt{1+\omega^2T_2^2}\ \tan^{-1}(\omega T_2)\right)\left(\sqrt{1+\omega^2T_3^2}\ \tan^{-1}(\omega T_3)\right)}$$

any constt has a phase angle of 0 → $\angle 0^\circ$
any term only having $j(\ )$ term → $\angle 90^\circ$
So, $k \rightarrow k\angle 0^\circ$
$j\omega \rightarrow \omega\angle 90^\circ$

$|a+jb| = \sqrt{a^2+b^2}$, $\arg(a,b) = \tan^{-1}(b/a)$

Magnitude $|G(j\omega)| = \frac{k\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}$

Phase angle $\angle G(j\omega) = \tan^{-1}(\omega T_1) - 90^\circ - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$

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Magnitude in dB = $20 \log |G(j\omega)|$

$$= 20 \log \left[\frac{k\sqrt{1+\omega^2 T_1^2}}{\omega\sqrt{1+\omega^2 T_2^2}\sqrt{1+\omega^2 T_3^2}} \right]$$

$$= 20 \log \left[\frac{k}{\omega} \times \sqrt{1+\omega^2 T_1^2} \times \frac{1}{\sqrt{1+\omega^2 T_2^2}} \times \frac{1}{\sqrt{1+\omega^2 T_3^2}} \right]$$

(Now, $\log ab = \log a + \log b$)

$$= 20 \log \frac{K}{\omega} + 20 \log \sqrt{1+\omega^2 T_1^2} + 20 \log \frac{1}{\sqrt{1+\omega^2 T_2^2}} + 20 \log \frac{1}{\sqrt{1+\omega^2 T_3^2}}$$

$$\Rightarrow \text{Magnitude} = 20 \log \frac{K}{\omega} + 20 \log \sqrt{1+\omega^2 T_1^2} - 20 \log \sqrt{1+\omega^2 T_2^2} - 20 \log \sqrt{1+\omega^2 T_3^2}$$

(So, when magnitude is expressed in dB, multiplication is converted to addition)

✱ BASIC FACTORS of $G(j\omega)$

FACTOR 1 → for Constt. Gain k

$G(s) = k$

$G(j\omega) = k = k\angle 0°$

Magnitude $= |G(j\omega)| = A|_{\text{in dB}} = 20 \log k$

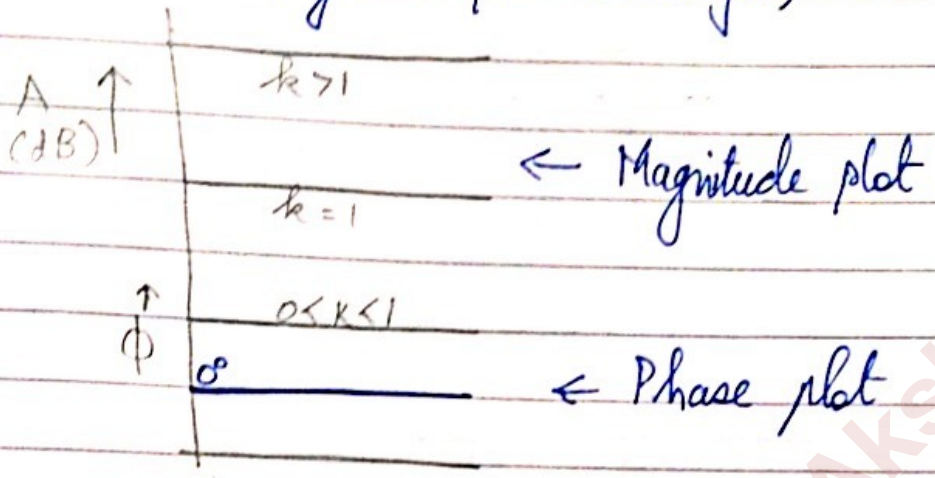
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Idea: $\frac{a}{jb} = \frac{a\angle 0}{b\angle 90^\circ} = \frac{a}{b}\angle 0-90^\circ$

$= \frac{a}{b}\angle -90^\circ$

Puffin
Date
Page

Phase angle $= \phi = \angle G(j\omega) = 0^\circ$



FACTOR 2 ⇝ Integral factor

$$G(s) = \frac{k}{s}$$

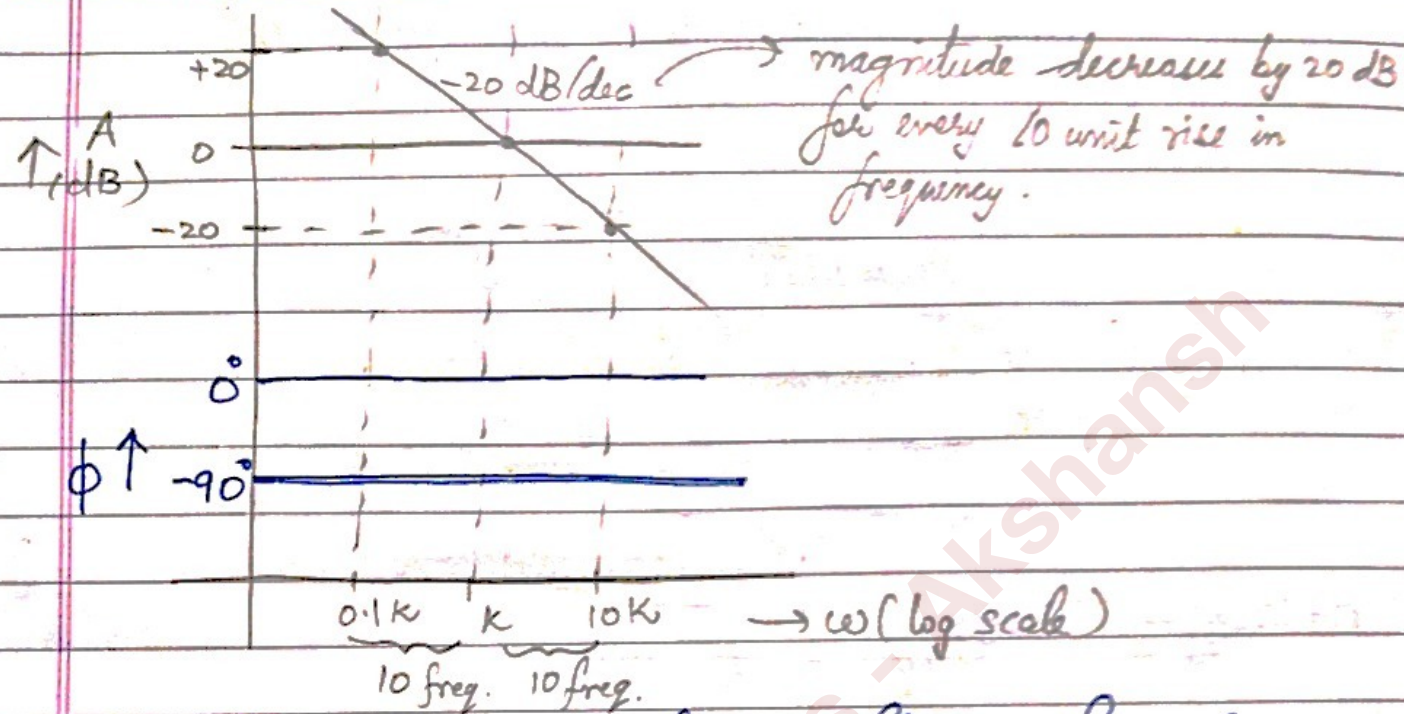
$$G(j\omega) = \frac{k}{j\omega} = \frac{k}{\omega}(\angle -90^\circ)$$

Magnitude, $A = |G(j\omega)|$ in dB $= 20\log\left(\frac{k}{\omega}\right)$

Phase angle, $\phi = \angle G(j\omega) = -90^\circ$

ω	A(dB)
$0.1k$	$20\log\left(\frac{k}{0.1k}\right) = 20$dB
K	0 dB
$10k$	$20\log\left(\frac{1}{10}\right) = -20$dB

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- Magnitude plot : st. line with a slope of -20 dB/dec passing through zero dB at $\omega = K$.
- Phase plot : st. line at $-90°$.

⇒ Subcase : Multiplicity

$$G(s) = \frac{k}{s^n}.$$

when an integral factor has multiplicity of 'n'

- Magnitude plot : St line with a slope of $-20n$ dB/dec. passing through zero dB at $\omega = k^{1/n}$
- Phase plot : st. line at $-90°n$

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FACTOR 3 ⇝ Derivative factor.

$$G(s) = ks$$
$$G(j\omega) = k j\omega$$
$$A = |G(j\omega)| \text{ in dB} = 20 \log(K\omega)$$
$$\phi = \angle G(j\omega) = +90^\circ$$

- Magnitude plot: ~~Derivative factor~~ St. line with a slope of +20 dB/dec. passing through zero dB at $\omega = 1/k$.
- Phase plot: st. line at $+90^\circ$.

⇒ Subcase: Multiplicity of n in derivative factor,

$$G(s) = ks^n$$
$$G(j\omega) = k(j\omega)^n$$

- Magnitude plot: St. line with a slope of $+20n$ dB/dec passing through zero dB at $\omega = \frac{1}{k^{1/n}}$
- Phase plot: st. line at $+90n^\circ$.

FACTOR 4 ⇝ First ~~order~~ order factor in denominator.

$$G(s) = \frac{1}{1+sT}$$

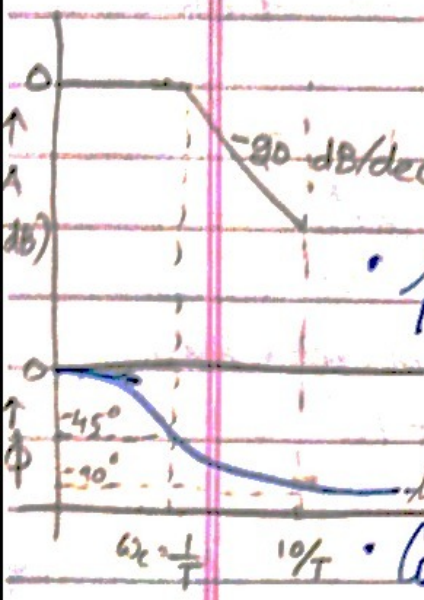
$$\Rightarrow G(j\omega) = \frac{1}{1+j\omega T} \quad ; G(j\omega): \text{Sinusoidal TF}$$

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$$A = |G_1(j\omega)| = 20 \log\left(\frac{1}{\sqrt{1+\omega^2 T^2}}\right)$$

$$= -20 \log\left(\sqrt{1+\omega^2 T^2}\right)$$

- Magnitude plot
  - one st. line at 0 dB for $0 < \omega < \frac{1}{T}$
  - another st. line with a slope of -20 dB/dec. for $\frac{1}{T} < \omega < \infty$.



- phase plot :

$$\Phi = -\tan^{-1}(\omega T)$$

ω	Φ
$1/T$	$-45°$
0	0
∞	$90°$

- Corner frequency / break frequency $\omega_c = \frac{1}{T}$ for the factor $\frac{1}{1+j\omega}$; divides the frequency response curve into 2 regions: one for low frequency, one for high frequency.

$\frac{1}{T}$: corner frequency / break frequency in $\frac{1}{1+j\omega T}$

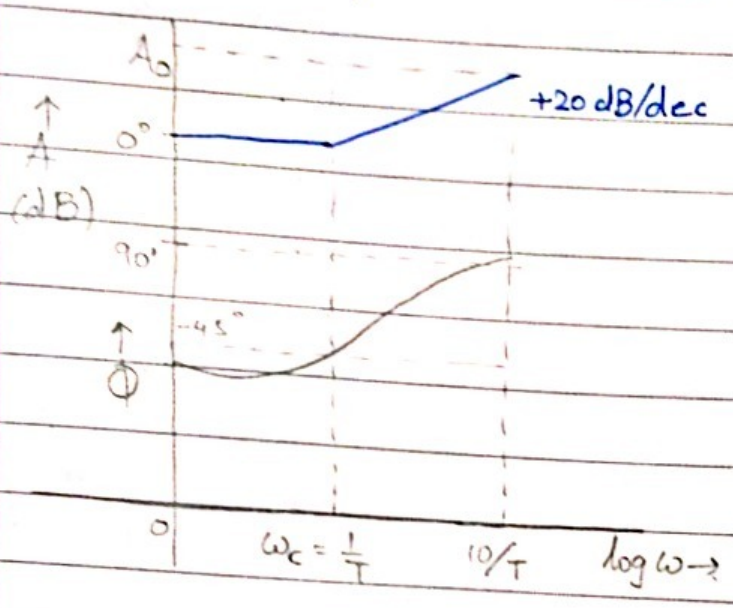
⇒ Subcase : When first order in denominator has a multiplicity of m.

$$G(s) = \frac{1}{(1+sT)^m}$$

$$A = -20m \log(1+\omega^2 T^2)$$

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FACTOR 5 ⇝ First order factor in the numerator of TF :



$G(s) = 1 + sT$

$G(j\omega) = 1 + j\omega T$

$A = 20 \log \sqrt{1 + \omega^2 T^2}$

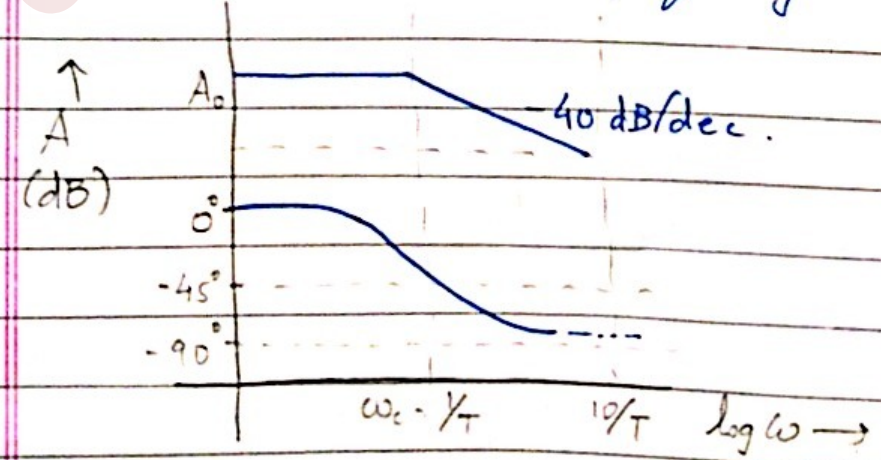
$\phi = \tan^{-1}(\omega T)$

⇒ Subcase : Multiplicity of order m :
- A changes by mA
- ϕ changes by $m\phi$.

FACTOR 6 ⇝ Quadratic factor in denominator.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(ω_c = natural frequency = ω_n)



$\omega_c = \omega_n$

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FACTOR 7 ⇝ Quadratic factor in numerator:
+ 40 dB/dec : Slope
(mirror image of denominator)
(factor 6)

Q. Sketch the Bode plot for the following TF. Determine the gain for gain crossover frequency (ω_{gc}) to be 5 rad/sec.

freq at which dbgain = 0
|gain| = 1

Given: $G(s) = \frac{k s^2}{(1+0.2s)(1+0.02s)}$

Sinusoidal TF = $G(j\omega)$

So, $G(j\omega) = \frac{k(j^2\omega^2)}{(1+0.2j\omega)(1+0.02j\omega)}$ → Term1 (derivative with order 2) → Term3

∵ k is unknown, assume it as 1 → Term2 ($\frac{1}{1+j\omega T}$; 1 order for $\omega_{c1} = \frac{1}{0.2}$)

So, $G(j\omega) = \frac{-k\omega^2}{(1+0.2j\omega)(1+0.02j\omega)}$

$= \frac{(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$

Since k is unknown, assume an initial value of 1.

(1) Magnitude plot :-

$\omega_{c1} = \frac{1}{0.2} = 5$ rad/s

$\omega_{c2} = \frac{1}{0.02} = 50$ rad/s

$(j\omega)^2$:- derivative factor w.r.t power of s

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Term	Corner freq. rad/sec	Slope dB/dec	Change in slope dB
$(j\omega)^2$	–	+40	
$\frac{1}{1+j0.2\omega}$	5	-20	$40+(-20)=20$
$\frac{1}{1+j(0.02)\omega}$	50	-20	$20+(-20)=0$

ascending order of ω_c

- Choose a low freq. (ω_L) s.t $\omega_L < \omega_{c1}$
- Choose a high freq. (ω_h) s.t $\omega_h > \omega_{c2}$

Let $\omega_L = 0.5$ rad/sec (< 5 rad/sec)

$\omega_h = 100$ rad/sec (> 50 rad/sec)

$A = |G(j\omega)|$ dB

→ at $\omega = \omega_L$ (0.5), $A = 20\log|(j\omega)^2| = 20\log(0.5)^2$
$= -12$ dB

→ at $\omega = \omega_{c1}$, $A = 20\log|(j\omega)^2| = 20\log(5)^2$
$= 28$ dB

- at low frequencies find magnitude only for numerator, because denominator vanishes.

↳ at $\omega = \omega_{c2}$, $A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \left(\log\frac{\omega_{c2}}{\omega_{c1}}\right)\right] + A(\omega = \omega_{c1})$

$= 20\log\frac{50}{5} + 28 = 48$ dB.

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↳ at $\omega = \omega_{h}$, $A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_{h} \times \log\left(\frac{\omega_h}{\omega_{c2}}\right) + A(\omega = \omega_{c2})\right]$

$= 0 \times \log\left(\frac{100}{50}\right) + 48 = 48$ dB

(2). Phase plot :-

$$\phi = \angle G(j\omega) = 180 - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega)$$

(a) rad/sec	$\tan^{-1}(0.2\omega)$ deg	$\tan^{-1}(0.02\omega)$ deg	ϕ	Point in
0.5	5.7	0.6	173.7 ≃ 174	e
1	11.3	1.1	168	f
5	45	5.7	130	g
10	63.4	11.3	106	h
50	84.3	45	50	i
100	87.1	63.4	30	j

↳ choose a scale of 1 unit = 20°
one y axis on RHS

Calculation of k :- Grain crossover freq. = 5 rad/s. (given)

Grain = 28 dB (from graph)

Grain crossover freq :- freq. in which db gain = 0.

So, seeing the graph, another lower plot is plotted, which has the pt 'b' giving gain zero (for ω = 5 rad/sec) → (how?)

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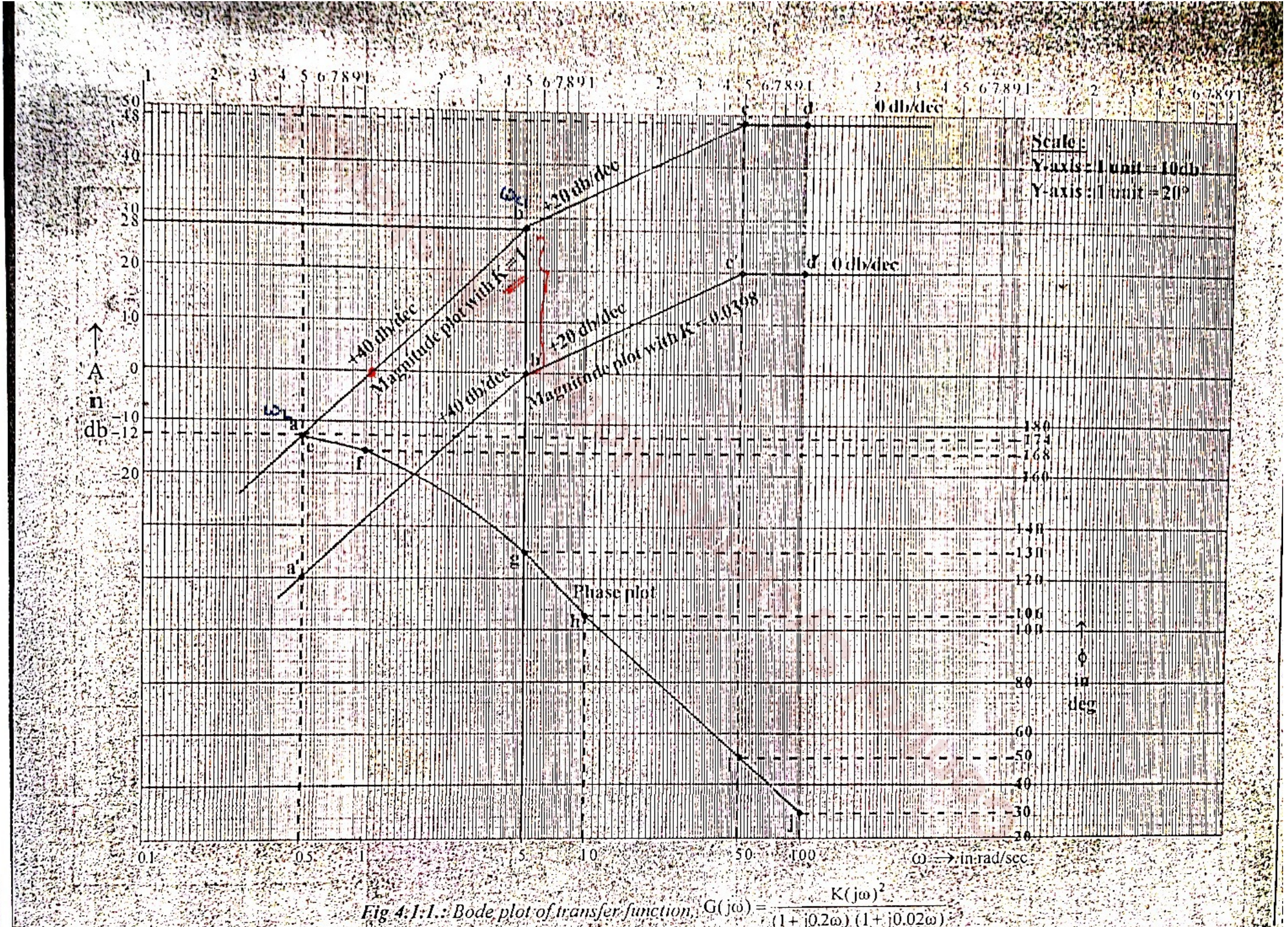


Fig. 4.1.1.: Bode plot of transfer function, $G(j\omega) = \dfrac{K(j\omega)^2}{(1 + j0.2\omega)\,(1 + j0.02\omega)}$

V. Imp ★ For any ques. in Bode plot, the TF should always be in time const form. (If in Pole Zero form, convert it).

- To every pt of magnitude plot, a dB gain of -28 dB should be added. It'll shift the plot downwards
- Magnitude correction is independent of frequency.

Gain, k is found by equating

$$20 \log K \text{ to } -28 \text{ dB}$$

$$\text{So, } \log K = \frac{-28}{20} \Rightarrow K = 0.0398$$

Ans

★ Determination of Gain Margin & Phase Margin.

- Gain margin in dB is ~~-ve~~ given by -ve of dB magnitude of $G(j\omega)$ at phase crossover frequency (ω_{pc}).
- ω_{pc} = freq. at which phase of $G(j\omega) = -180°$.
- ϕ_{gc} : ϕ angle of $G(j\omega)$ at gain crossover frequency ω_{gc}.
- ω_{gc} : freq. at which dB magnitude of $G(j\omega)$ is zero.
- Phase margin $\gamma = 180 + \phi_{gc}$.

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Gain margin

Phase margin

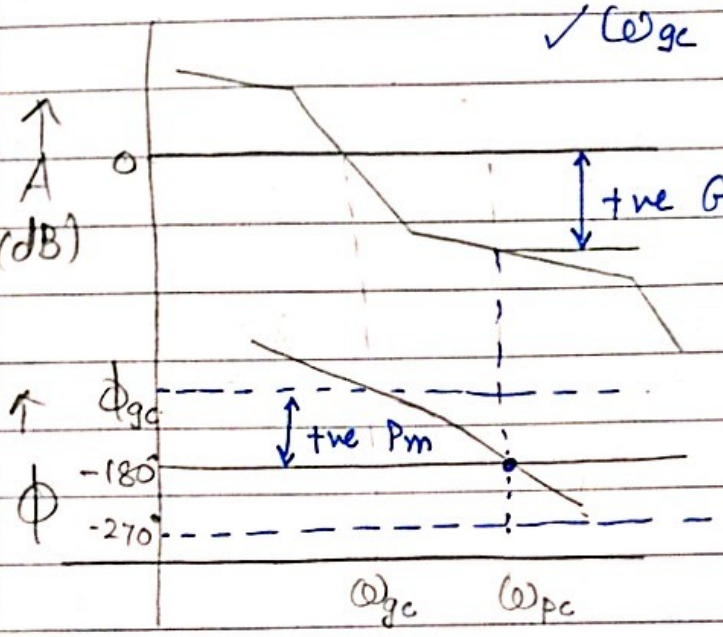
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For stability, ✓ (GM) & (PM) : +ve

✓ $\omega_{gc} < \omega_{pc}$



Q. Plot Bode diagram for following TF & obtain gain & phase crossover frequency

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$$

&,

$$G(j\omega) = \frac{10}{(j\omega)(1+j0.4\omega)(1+j0.1\omega)}$$

- Magnitude plot : $\omega_{c1} = \frac{1}{0.4} = 2.5$ rad/sec

$\omega_{c2} = \frac{1}{0.1} = 10$ rad/sec

corner frequency

Control Systems Notes - Akshansh

Term	Corner freq. rad/sec	Slope dB/dec	Change in slope
$10/j\omega$	—	-20	—
$1/1+0.4j\omega$	$\omega_{c1} = 2.5$	-20	$-20-20 = -40$
$1/1+0.1j\omega$	$\omega_{c2} = 10$	-20	$-40-20 = -60$.

req of for Phase plot

Phase angle, $\phi = -90^\circ - \tan^{-1}(0.4\omega) - \tan^{-1}(0.1\omega)$

→ Phase angle for $\frac{10}{j\omega} = -90^\circ$.

→ Phase angle for $\left(\frac{1}{1+0.4j\omega}\right) = -$ Phase angle $(1+0.4j\omega)$

$= -\tan^{-1}(0.4\omega)$

let $\omega_L = 0.1$ rad/sec $(< \omega_{c1})$
& $\omega_h = 50$ rad/sec $(> \omega_{c2})$

$A = |G(j\omega)|$ in dB.

A, at

→ $\omega = \omega_L = 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \frac{10}{0.1} = 40$ dB.

→ $\omega = \omega_{c1} = 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \frac{10}{2.5} = 12$ dB.

→ $\omega = \omega_{c2}$; $= \left[(\text{Slope from } \omega_{c1} \text{ to } \omega_{c2}) \times \left(\log \frac{\omega_{c2}}{\omega_{c1}}\right) + A\big|_{\text{at } \omega = \omega_{c1}}\right]$

$= -40 \times \log \frac{10}{2.5} + 12 = -12$ dB.

Control Systems Notes - Akshansh

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↳ at $\omega = \omega_h$; $A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \left(\log \frac{\omega_h}{\omega_{c2}} \right) \right] + A\big|_{\text{at } \omega = \omega_{c2}}$

$= -60 + \log \frac{50}{10} + (-12) = -54\,dB$

- Phase plot :-

Phase angle of $G(j\omega)$ as a fn of ω is given by

using it ↓

$$\phi = -90 - \tan^{-1}(0.4\omega) - \tan^{-1}(0.1\omega)$$

ω (rad/sec)	ϕ	Pts. in ϕ plot
0.1	-92	e
1	-118	f
2.5	-150	g
4	-170	h
10	-210	i
20	-286	j

Ans ↳ $\omega_{gc} = \omega_{pc} = 5$ rad/sec.

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POLAR PLOT

FREQUENCY RESPONSE ANALYSIS

Bode Plot Polar Plot Nyquist Plot

Polar Plot:

of a sinusoidal TF is defined as plot b/w $G(j\omega)$ & $\angle G(j\omega)$ as ω is varied from 0 to ∞

- polar plot: locus of vectors $|G(j\omega)| \angle G(j\omega)$
- negative ϕ angle measured clockwise.
  +ve ϕ angle measured anticlockwise.

- Magnitude, $A = |G(j\omega)\,H(j\omega)|$
- Phase angle, $\phi = \angle G(j\omega)\,H(j\omega)$

} as ω varies from 0 to ∞.

SKETCHING POLAR PLOT:

(in Bode plot, corner frequencies were chosen)

Step 1: Determine TF, $G(s)$ of a sys.

S2) $s = j\omega$ in $G(s)$ to obtain $G(j\omega)$.

S3) Obtain expressions for $|G(j\omega)|$ & $\angle G(j\omega)$

S4) Calculate $|G(j\omega)|$ & $\angle G(j\omega)$ for $\omega = 0$ & (to) $\omega \to \infty$ (in tabular form).

($\omega = 0$ & $\omega = \infty$ to be included/not, it'll be told in ques.)

- Concentric circles chart is given to plot.

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Type zero sys. with OLTF (order 1) given as :-

$$G(s)H(s) = \frac{1}{Ts+1} \quad (T \cdot \text{const}^n)$$

S2) $s \to j\omega$

$$\Rightarrow G(j\omega)H(j\omega) = \frac{1}{T(j\omega)+1} = \frac{1+j(0)}{1+j(T\omega)}$$

S3) $$|G(j\omega)H(j\omega)| = A = \frac{1}{\sqrt{1+\omega^2T^2}}$$

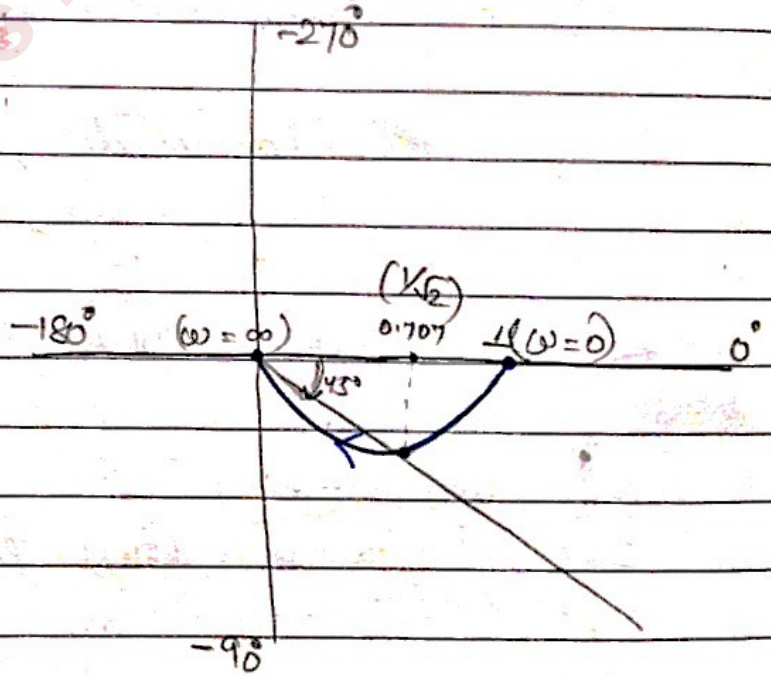
$$\angle G(j\omega)H(j\omega) = \phi = \frac{\tan^{-1}\left(\frac{0}{1}\right)}{\tan^{-1}(\omega T)} = \frac{0^\circ}{\tan^{-1}(\omega T)}$$

$$\Rightarrow \phi = -\tan^{-1}(\omega T)$$

S4)

ω	A	ϕ
0	1	0
$\frac{1}{T}$	$1/\sqrt{2}$	-45°
$\frac{10}{T}$	$\frac{1}{\sqrt{101}}$	-84.2°
∞	0	-90°

Table 1



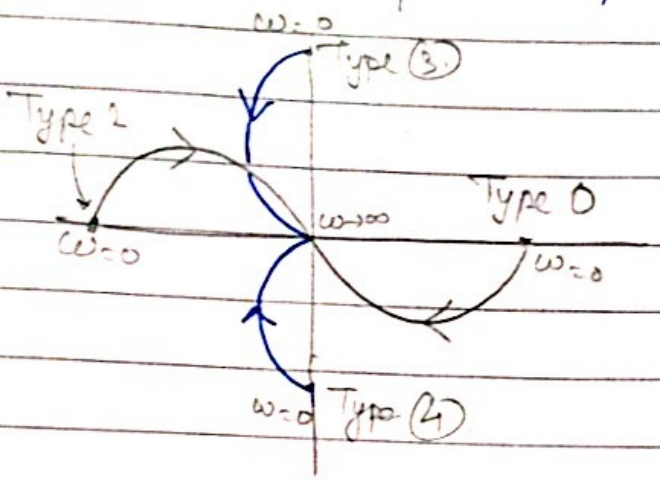
Plotting done using (3 pts) & joining them

1st 2 rows & last row of Table 1

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Idea :- Start of Polar plot



TYPE : decides start of polar plot
ORDER : decides end of polar plot

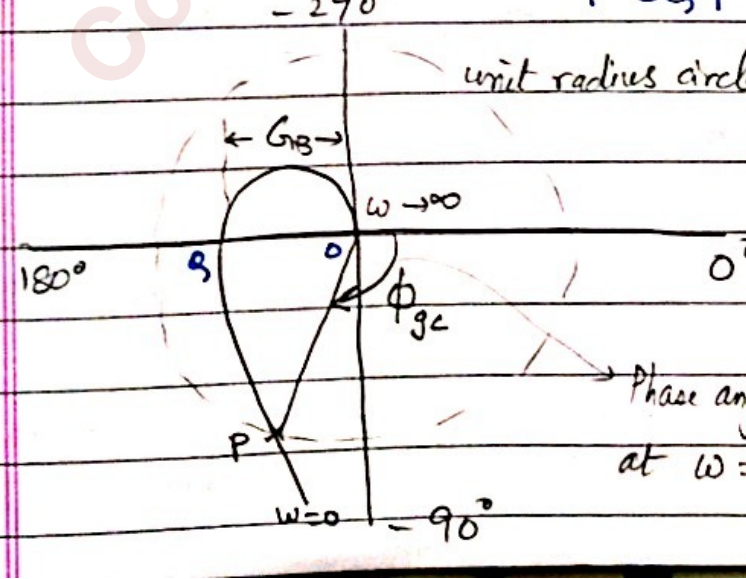
✱ GAIN MARGIN & PHASE MARGIN from POLAR PLOT

Gain Margin (G_m)

The margin in gain which can be increased till the sys reaches the verge of instability. For a stable sys, both G_m & P_m should be +ve OR gain crossover frequency should be less than Phase crossover frequency.

Stable sys
G_m & $P_m > 0$
or
$\omega_{gc} < \omega_{pc}$

$$GM = 20 \log_{10} \left| \frac{1}{OQ} \right| dB.$$



$$G_m = K_g = \frac{1}{|G_1(j\omega_{pc})|}$$

Control Systems Notes Akshansh

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Q. Given :- OLTF of a unity feedback sys :-

$$G(s) = \frac{1}{s(s+1)(2s+1)} \rightarrow \text{in } 1+Ts \text{ form.}$$

Sketch polar plot, determine G_m & P_m.

S2) $$G(j\omega) = \frac{1}{(j\omega)(1+j\omega)(1+2j\omega)}$$

S3) $$|G(j\omega)| = \frac{1}{(\omega)\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$$

$$\angle G(j\omega) = \frac{\tan^{-1}\left(\frac{0}{1}\right)}{\tan^{-1}\left(\frac{\omega}{0}\right) + \tan^{-1}\left(\frac{\omega}{1}\right) + \tan^{-1}\left(\frac{2\omega}{1}\right)}$$

$$= 0 - 90^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$$

S4) Corner frequency,

$$\omega_{c1} = \frac{1}{2} = 0.5 \text{ rad/s}.$$

$$\omega_{c2} = \frac{1}{1} = 1 \text{ rad/sec}.$$

* Choice of freq. for plotting polar plot are taken around corner frequencies.

ω (rad/sec)	0	0.2	0.3	0.4	ω_{c1} 0.5	0.7	0.9	ω_{c2} 1	∞
$\lvert G(j\omega)\rvert$	∞	4.55	2.73	1.81	1.26	0.68	0.4	0.36	0
$\angle G(j\omega)$	-90°	-123°	-137.76	-150.4	-161.5	-179.4°	-192.9	-198.4°	-270°

Control Systems Notes - Akshansh

• Assumptions in polar plot: by convention, ~~-ve~~
-ve ϕ angles : measured in clockwise dir^n
+ve ϕ angles : measured in anti-clockwise dir^n.

* Values of ω: To be taken ourselves
(If not told in ques.)

See the graph for the plot

Finding Gain margin
See the pt. where plot crosses x axis
Find ω at that pt

Phase margin.
The plot crosses unity circle at a pt. From 0 to that pt, the clockwise dir^n gives the phase margin (ϕ_{pc}).

---

eg 2 $G(s) = \dfrac{k}{s(1+0.5s)(1+4s)}$, $H(s) = 1$

Sketch polar plot & determine the value of k so that (i) GM is 20 dB.
(ii) PM is 30°.

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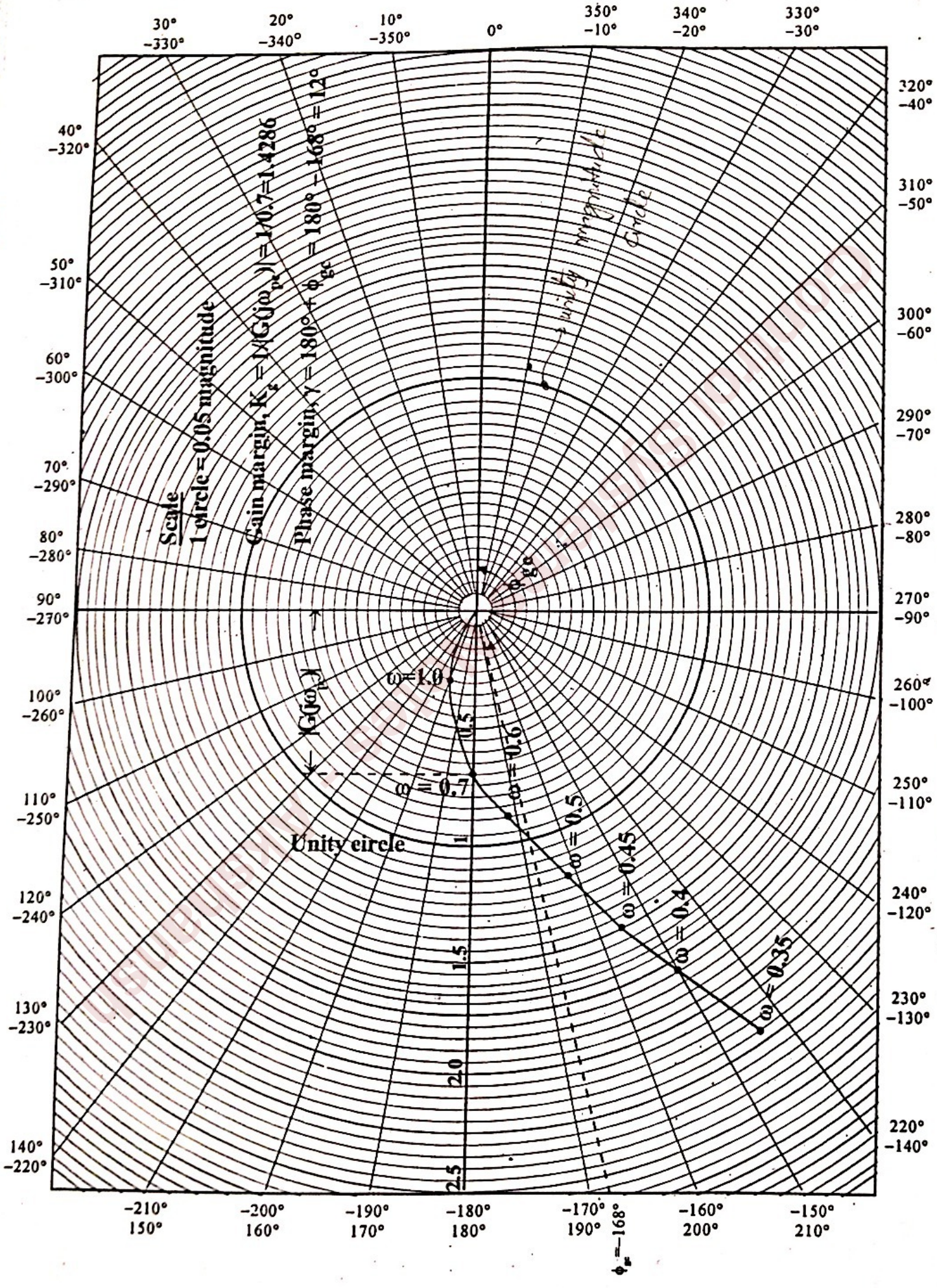


Fig 4.7.1: Polar plot of $G(j\omega) = 1/j\omega\,(1+j\omega)\,(1+j2\omega)$ (using polar coordinates).

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s2) $G(j\omega) = \frac{k}{(j\omega)(1+0.5j\omega)(1+4j\omega)}$

s3) $|G(j\omega)| = A = \frac{1}{(\omega)\sqrt{1+0.5^2\omega^2}\sqrt{1+16\omega^2}}$ → Assuming k = 1 initially

$\angle G(j\omega) = -\tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}(0.5\omega) - \tan^{-1}(4\omega)$

$\Rightarrow \phi = -90° - \tan^{-1}(0.5\omega) - \tan^{-1}(4\omega)$

Corner frequency —

$\omega_{c_1} = \frac{1}{0.5}$, $\omega_{c_2} = \frac{1}{4}$

$\Rightarrow \omega_{c_1} = 2$, $\omega_{c_2} = 0.25$

s4)

ω	$\lvert G(j\omega)\rvert$ (A)	$\angle G(j\omega)$ (ϕ)
0	∞	$-90°$
ω_{c_2} 0.25	2.82	$-142.12°$
0.5	0.701	$-167.47°$
0.8	0.346	$-184.44°$
1	0.2169	$-192.52°$
ω_{c_1} 2	0.043	$-217.87°$
∞	0	$-270°$

A plot b/w A & ϕ

(ω: not there in the plot)

Now start plotting
On the polar graph, dark circle : unity magnitude
centre ⇒ $\omega = 0$, & then, each division = 0.1 (under std. scale)
(The scale can also be changed to expand graph)

Control Systems Notes - Akshansh

* For GM, see the ω_{pc} → phase crossover frequency
* For PM, see the ω_{gc} → gain crossover

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* The drawn polar plot cuts the unity magnitude circle at pt. B
The frequency corresponding to this is called gain crossover frequency, ω_{gc}.
The phase angle, $\phi_{gc} = -163°$.

$$PM\Big|_{K=1} = 180° + \phi_{gc} = 17°$$

The polar ~~plot~~ plot cuts the $-180°$ line at a magnitude of 0.45.

Now, $GM = \frac{1}{0.45} = 2.22$.

in dB, $= -20 \log(2.22)$

(i) Given, GM = 20 dB,

So, $20 \log \frac{1}{G_A} = 20\,dB \Rightarrow G_A = 0.1$

G_A → magnitude corresponding to new GM.

$$K = \frac{G_A}{G_B} = \frac{0.1}{0.45} = 0.22$$

[for $G_B = 0.45$, $K = 1$
for $G_A = 0.1$, $K = \frac{1}{0.45} \times 0.1$]

(ii) $\gamma = 30° = 180° + \phi_{gc}$

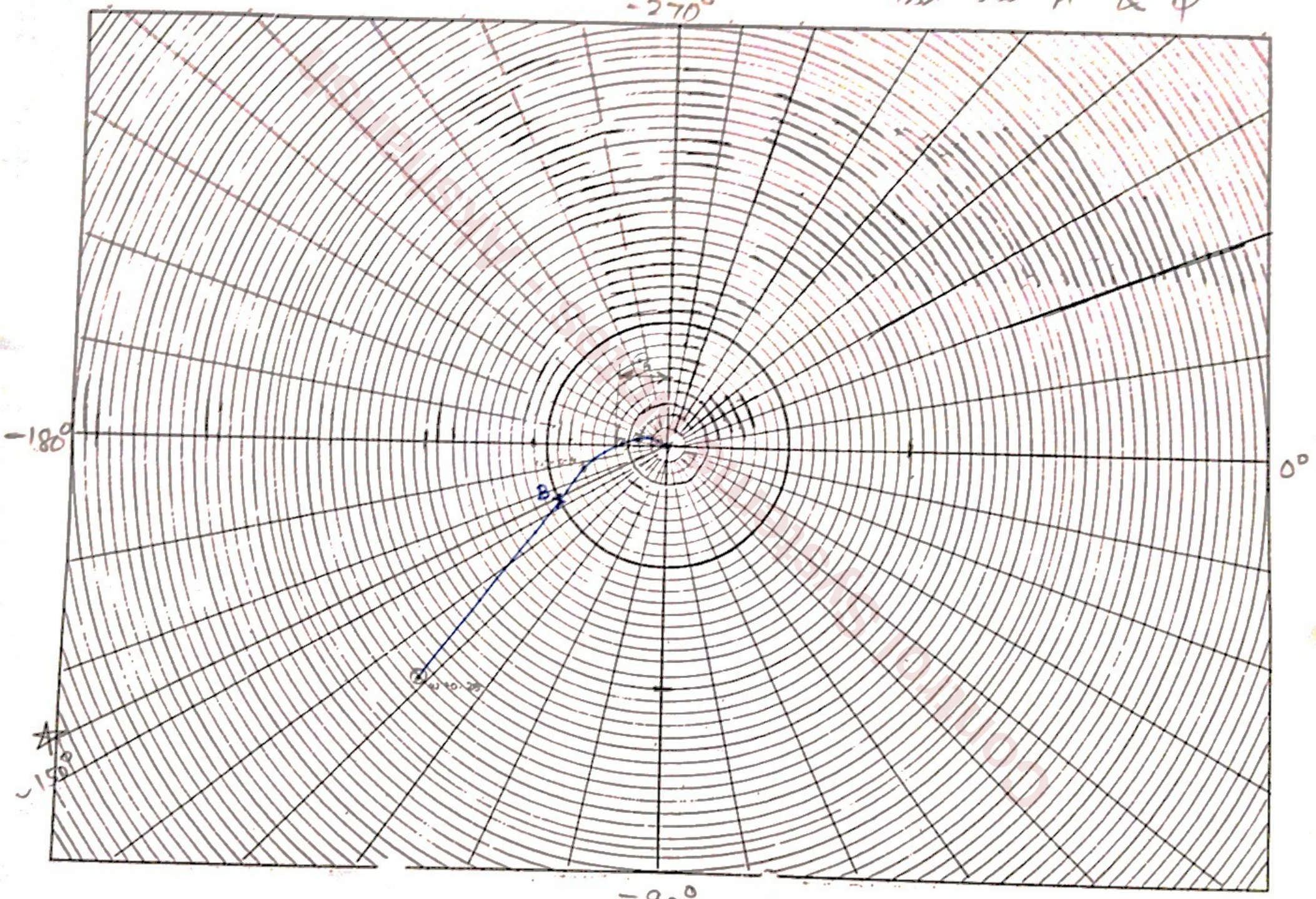
$\Rightarrow \phi_{gc} = -150°$ (line cuts graph at 2.2)

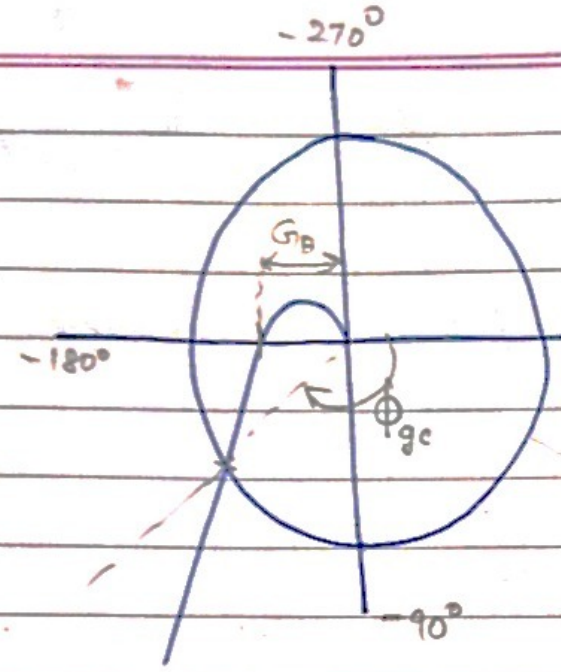
$$K = \frac{1}{2.2} = 0.45$$

[* Plot cuts unity circle at pt B
Magnitude = 1]

Control Systems Notes - Akshansh

Plot V_{in} Λ & ϕ





$$GM = \frac{1}{G_B}$$

$$PM = 180 + \phi_{gc}$$

Problem (AA) on BODE PLOT

Sketch the Bode Plot for the following TF & determine GM & PM.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$$

↳ ∃ a quadratic factor

~~S2)~~ $$G(j\omega) = \frac{75(1+0.2s)}{(j\omega)(-\omega^2+16j\omega+100)}$$

first convert to time const^n form.

$s^2+16s+100 \equiv s^2+2\zeta\omega_n s+\omega_n^2$

$\omega_n^2 = 100 \Rightarrow \omega_n = 10$ rad/s

& $2\zeta\omega_n = 16 \Rightarrow \zeta = 0.8$

↳ $0<\zeta<1$

↳ underdamped

$$G(s) = \frac{75(1+0.25)}{s(100)\left[\frac{s^2}{100}+\frac{16}{100}s+1\right]}$$

Control Systems Notes - Akshansh

$$\Rightarrow G(s) = \frac{0.75(1+0.2s)}{s(1+0.01s^2+0.16s)}$$

Now

$$G(j\omega) = \frac{0.75(1+0.2j\omega)}{(j\omega)(1-0.01\omega^2+0.16j\omega)}$$

Seeing corner frequencies

↳ $\omega_{c1} = \frac{1}{0.2} = 5$ rad/s → for numerator

ω_{c2} = corner frequency for quadratic factor = ω_n

$\Rightarrow \omega_{c2} = 10$

Now,

Assuming values for ω_l & ω_h

ω_l → 0.5 rad/s ($< \omega_{c1}$)

ω_h → 20 rad/s ($> \omega_{c2}$)

Now

$A = |G(j\omega)|$ (in dB)

Term	Corner freq (rad/s)	Slope (dB/dec)	Change in Slope (dB/dec)
Integral factor (ω: denominator) $\frac{0.75}{\omega}$	—	−20	—
1st order factor in numerator $1+0.2j\omega$	5	+20	−20+20 = 0
$1-0.01\omega^2+j0.16\omega$	10	−40	0−40 = −40

10 → fixed value for quadratic factor in denominator

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* Phase plot cuts 0 dB line :- PM

Now

$$\phi = \angle G(j\omega) = 0 + \tan^{-1}\left(\frac{0.2\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right)$$

(for $\omega \le \omega_n$; ω for $j\omega$)

$$= \tan^{-1}(0.2\omega) - 90^\circ - \tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right)$$

Plot

(ω, A) ↑

$(0.5, 3.5)$ pt A

Now, $A\big|_{\omega=\omega_1} = 20\log\left|\frac{0.75}{0.5}\right| = +3.5\ dB$

$(5, -16.5)$ pt B $\quad A\big|_{\omega=\omega_{c1}} = 20\log\left|\frac{0.75}{5}\right| = -16.5\ dB$

$(10, -16.5)$ pt C $\quad A\big|_{\omega=\omega_{c2}} = 0 \times \log\left|\frac{\omega_{c2}}{\omega_{c1}}\right| + (-16.5) = -16.5$

($\omega_{c2} = 10$, $\omega_{c1} = 5$; for quadratic)

$(20, -28.5)$ pt D $\quad A\big|_{\omega=\omega_h} = [-40 \times \log(\frac{20}{10})] + (-16.5) = -28.5\ dB$

Now,

ω	ϕ	Pt.	
0.5	−88	a	eq^n for ϕ can vary only till $\omega \le \omega_n$ i.e. $\omega \le 10$
1	−88	b	
5	−92	c	
10	−116	d	
20	−147	e	for $\omega > 10$
50	−168	f	
100	−174	g	

See ↗

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See 0 dB line intersecting magnitude graph.
The value of ϕ at that pt. is ϕ_{gc}.
So. PM = $\gamma = 180 - \phi_{gc}$.

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Ans :- PM = 100°

GM → ∞ ← If Phase plot not touching -180°, then, the value of GM = $\frac{1}{0} = \infty$

Note

$$\phi = \tan^{-1}(0.2\omega) = 90^\circ - \tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right)$$

↳ valid for $\omega \leq \omega_n$
$\Rightarrow \omega \leq 10$.

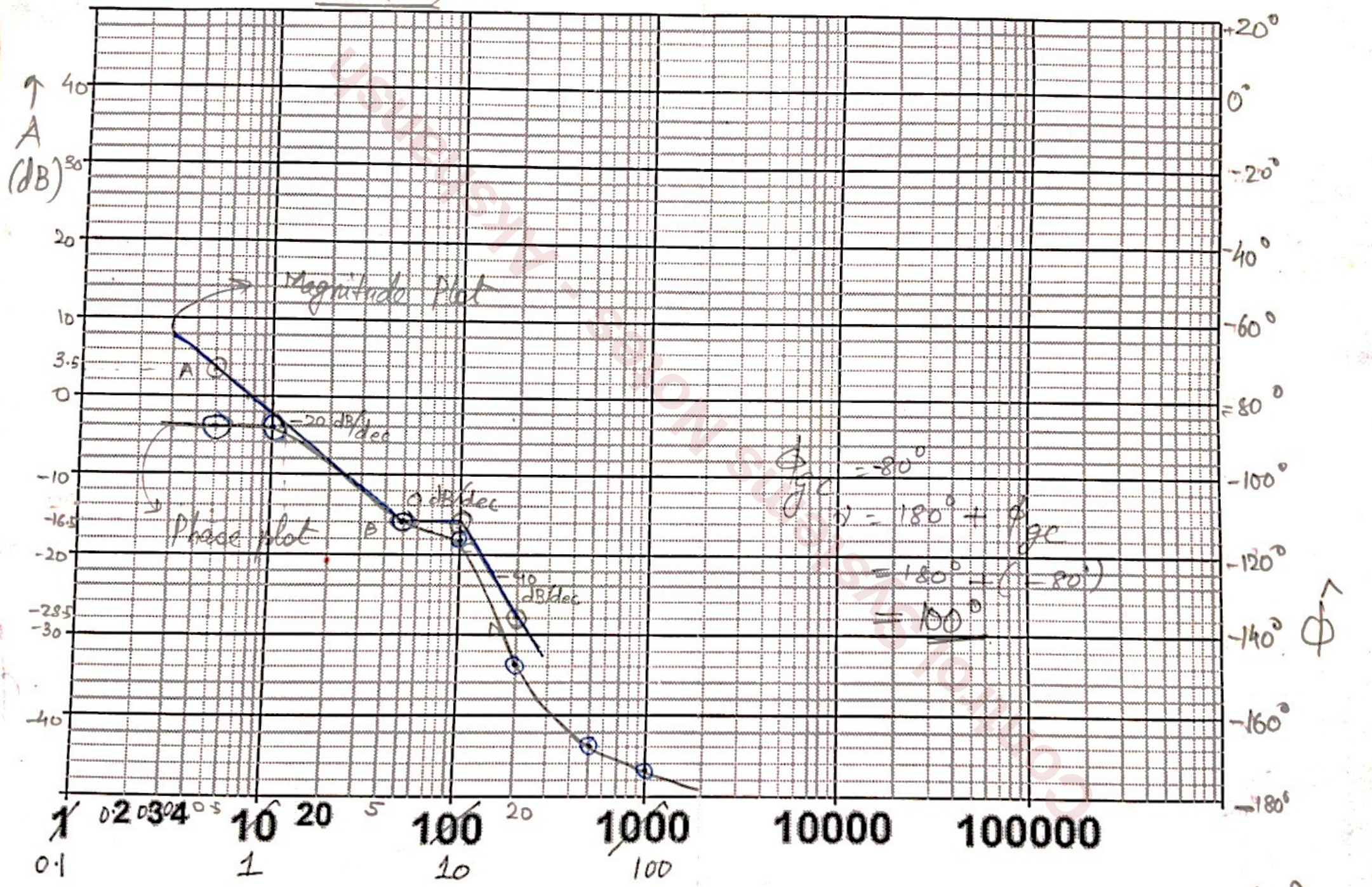
ϕ changes to

for $\omega > \omega_n$
$$= \tan^{-1}(0.2\omega) - 90^\circ - \tan^{-1}\left[\left(\frac{0.16\omega}{1-0.01\omega^2}\right) + 180^\circ\right]$$

↳ need to be done only if ∃ quadratic factor in numerator or denominator.

Control Systems Notes - Akshansh

PROBLEM (AA)


A (dB)
40
30
20
10
3.5
0
-10
-16.5
-20
-28.5
-30
-40
+20°
0°
-20°
-40°
-60°
-80°
-100°
-120°
-140°
-160°
-180°
Magnitude Plot
Phase plot
A
B
-20 dB/dec
0 dB/dec
-40 dB/dec
1
0.2
0.3
0.4
0.5
10
20
5
100
20
1000
10000
100000
0.1
1
10
100
ω (rad/sec)
ϕ
Lead in ω_gc = -80°
ψ = 180° + ϕ_gc
= 180° + (-80°)
= 100°

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STATE SPACE ANALYSIS

* Dealing with MIMO sys (Multiple i/p, Multiple o/p sys).
* In such a sys, ∃ state variables (complexity reduces).

Taking Y (as o/p) & U (as i/p), some ways of represent^n of such a sys is by :

(a) CANONICAL form :-

(1) Matrix arrangement is seen in CONTROLLABLE form
(2) Observable Canonical form : Transpose of Controllable canonical form
(3) Diagonal canonical form : Suitable for the sys having distinct roots
↳ the distinct roots form the diagonal of the matrix.

Q. Consider a sys. given by $\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$

Obtain state space represent^{ns} in the controllable canonical form, observable & diagonal canonical forms.

The generic form for Canonical form is written as :-

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots}$$

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For our case, $n = 2$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s^1 + b_2}{s^2 + a_1 s + a_2}$$

Comparing,

$b_0 = 0,\ b_1 = 1,\ b_2 = 3$

$a_1 = 3,\ a_2 = 2$

Controllable canonical

Std. form — State variable matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

for us $= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

So

Std. form — O/P Matrix

$$y = \begin{bmatrix} b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$$

$$= \begin{bmatrix} 3 - (2 \times 0) & 1 - (3 \times 0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$

$$\Rightarrow y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observable Canonical (Transpose of Controllable)

State variable matrix $= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$

$$= \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u$$

& $y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (\because b_0 = 0)$$

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Theory: Std. form of state variable matrix in Controllable form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

Std. form of o/p matrix in Controllable form

$$y = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Diagonal canonical form:-

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}.$$

The denominator polynomial has distinct roots

i.e, $s^2+3s+2 = (s+2)(s+1)$

So, $\frac{Y(s)}{U(s)} = \frac{s+3}{(s+1)(s+2)}$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2}.$$

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By partial fraction,

$$s+3 = c_1(s+p_2) + c_2(s+p_1) \quad (\because b_0 = 0)$$
$$= c_1(s+2) + c_2(s+1)$$

Comparing coeff. of $s \Rightarrow 1 = c_1 + c_2$.

Const term $\Rightarrow 3 = 2c_1 + c_2$.

$$\Rightarrow c_1 = 2, \; c_2 = -1$$

So, diagonal form is given as;-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -p_1 & 0 \\ 0 & -p_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

$$\& \quad y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u.$$

Hence,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

$$y = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (0)u.$$

$$\Rightarrow y = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note :- Controllable, Observable & Diagonal Canonical forms have a std. way of writing the
- state variable matrix
- o/p matrix.

Using these std. forms, they can be evaluated.

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☆ CONTROLLER

→ It used to modify the error signal & to achieve better control action.
→ It will modify transient response & steady state error of the system.

☆ Types of Controller

→ 1) P Controller : Proportional Controller.
2) PI Controller : Proportional & Integral Controller
3) PD " : Proportional & Derivative.
4) PID " : Proportional Integral Derivative.

1) 'P' Controller:

produces an o/p signal which is proportional to error signal. i.e.,

$$u(t) \propto e(t) \quad \rightarrow \text{proportional}$$

$$= K_p\, e(t)$$

$$\Rightarrow U(s) = K_p\, E(s)$$

$$\text{So, } \frac{U(s)}{E(s)} = K_p = TF = \text{gain}.$$

2) 'PI' Controller:

produces an o/p signal having 2 terms:
(i) proportional to error signal.
(ii) proportional to integral of error signal.

So,

$$u(t) \propto \left[e(t) + \int e(t)\,dt \right]$$

$$= K_p \left[e(t) \right] + \left[\frac{K_p}{T_i} \int e(t)\,dt \right]$$

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$$\Rightarrow u(t) = K_p e(t) + \frac{K_p}{T_i}\int e(t)dt$$

↳ T_i : Integral time.

Hence, $U(s) = K_p E(s) + \frac{K_p}{T_i}\left(\frac{E(s)}{s}\right)$

Hence, $\frac{U(s)}{E(s)} = K_p\left[1 + \frac{1}{T_i s}\right]$

$= K_p\left[\frac{T_i s + 1}{T_i s}\right]$

3) 'PD' Controller:

Produces an o/p signal having 2 terms:

(i) proportional to error signal

(ii) proportional to derivative of error signal.

$$u(t) \propto e(t) + \frac{d}{dt}e(t)$$

$$= K_p e(t) + K_p T_d \frac{d}{dt}e(t)$$

$$\Rightarrow U(s) = K_p E(s) + K_p T_d s E(s)$$

$$\& \frac{U(s)}{E(s)} = K_p + K_p T_d s. = K_p[1 + T_d s]$$

4) 'PID' Controller: 3 terms :-

$$u(t) \propto e(t) + \int e(t)dt + \frac{d}{dt}e(t)$$

$$= K_p e(t) + \frac{K_p}{T_i}\int e(t)dt + K_p T_d \frac{d}{dt}e(t)$$

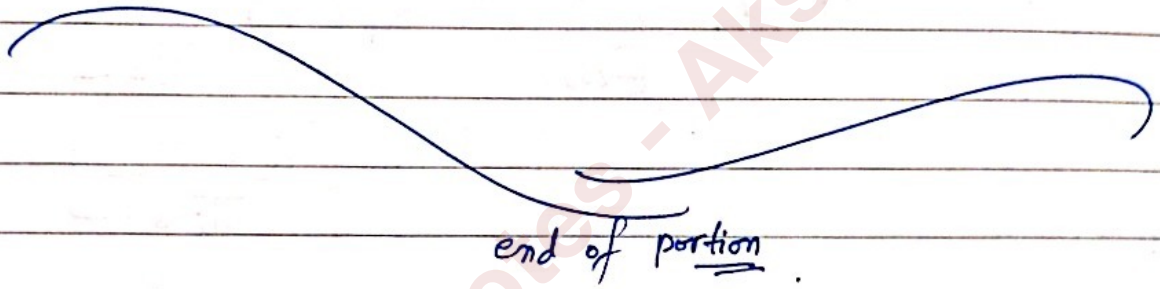
$$\Rightarrow U(s) = K_p E(s) + \frac{K_p}{T_i}\frac{E(s)}{s} + K_p T_d s E(s)$$

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So, $\frac{U(s)}{E(s)} = K_p + \frac{K_p}{T_i s} + K_p T_d s$

$= K_p \left[1 + \frac{1}{T_i s} + T_d s \right]$



end of portion.

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